STUDENTS AND TEACHERS USING A DIGITAL CURRICULUM AND PEDAGOGY IN SECONDARY MATHEMATICS

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Scott Bates Farrar

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CERTIFICATION OF APPROVAL

I certify that I have read Students and Teachers Using a Digital Curriculum and Pedagogy in Secondary Mathematics by Scott Bates Farrar, and that in my opinion this work meets the criteria for approving a field study submitted in partial fulfillment of the requirements for the degree: Master of Arts in Education: Mathematics Education at San Francisco State University.

Judith Kysh, Professor, Secondary Education and Mathematics

Maria Zavala, Assistant Professor, Elementary Education

Abstract

The digital age offers a chance to rethink mathematics curriculum and pedagogy. In this paper, I review research on the capabilities afforded by dynamic mathematics software, and study a case of a classroom using two types of mathematics learning software. Assessment Mathematics Software (AMS) provides exercises for students and will grade the results. Dynamic Mathematics Software (DMS) attempts to interpret student inputs into a mathematical model and will respond mathematically. AMS encouraged procedural discourse while DMS encouraged exploratory and justification discourse from both student and teacher. DMS may work best with significant preparation from the teacher while AMS may work best as an external individual support.

Keywords: Dynamic Geometry Software, Dynamic Mathematics Software, GeoGebra, Desmos, Digital Curriculum, Khan Academy, discourse, agency, professional development, curriculum, pedagogy

I certify that the Abstract is a correct representation of the content of this field study.

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Judith Kysh, Committee Chair Date

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Chapter 1: Introduction

Computers offer powerful capabilities for teaching and learning. In the past few decades, computers have been integrated sporadically into secondary mathematics classrooms as tools for teaching and learning. Examples include arithmetic and scientific/graphing calculators, exercise generating and answer checking software, computer algebra systems, structured programming environments, video tutoring, interactive simulations, and dynamic geometry software. However, in this same time period, secondary math curricula have retained the style of textbooks and worksheets. Pedagogies that more fully use technology have not been widely seen. Researchers ask for more careful study about learning via technology and the unexplored details therein. So far, "[curriculum developers'] design and approaches reflect content and suggest pedagogies more suited to traditional curricula" (Clements, Sarama, Yelland, & Glass, 2008). Educators can and should re-conceptualize curricula and pedagogy to take advantage of capabilities computer software affords, and begin to study what benefits and tradeoffs these digital practices and materials offer (Clements et al., 2008; diSessa, 2007; Harel, 2013; Johnson, 2016). What are the differing characteristics of current mathematics software? In what ways do software elements influence teacher pedagogy and student discourse? And how do students conceptualize learning when using mathematics software?

Education on computers is almost as old as the computer itself. But in the past decade the symbiotic growth of the Internet and of internet-connected devices (such as smartphones, tablets, laptops) has meant that education technology can be based in software or data transferred over the internet. The ease of the internet for distribution potentially forms a wide foundation of student access upon which educators can implement software-based interactive tools. Interactivity with the computer potentially revises the representational basis for thinking and learning. “Students can think and learn with these in substantially different ways, leading to new possibilities for curricular scope and sequence” (diSessa, 2007). Non-interactive uses of computers— such as watching a video lecture, reading text, or doing computer generated practice— are reformulations of non-computerized pedagogy (Clements et al., 2008). For the purpose of this paper I define *interactivity* as the computer software that responds immediately to user inputs by providing feedback or analysis. I categorize interactive elements of mathematics education software as either: Dynamic Mathematics Software “DMS” or Assessment Mathematics Software “AMS”.

DMS attempts to model mathematical concepts for the purpose of interaction and investigation. Martin (2016) describes this as mathematical feedback. A graph of a parabola that changes according to user-alterable parameters is an example of DMS. In contrast, AMS attempts to assess the learner’s understanding of mathematical concepts and communicate that assessment to the learner or another party like a parent or teacher. For example, AMS might ask a user to solve exercises about a parabola, analyze the inputted solutions to determine what the user understands, and provide feedback based on that analysis. The feedback may tell the users if they were right or wrong, attempt to direct the user to resources for help or to fix a misunderstanding. AMS may also alter the content or difficulty level of future exercises. AMS does not attempt to model or interactively represent the mathematical concept, and DMS does not assess the user’s inputs for accuracy or achievement. A single software product may contain elements of both

AMS and DMS. This paper explores existing typical DMS and AMS, their use in classroom curriculum and the pedagogy involved in their implementation.

DMS is intended for either learner or teacher use. It allows creation or manipulation of objects on the screen that contain mathematical properties. These objects could be purely geometric like circles, analytic like functions, data-based like charts and spreadsheets, they may model a mathematical concept as a game. In each of these cases, the user receives feedback information from the software such as dependent objects changing size or the output of an algebraic request. The word *dynamic* pertains to the software responding to inputs and providing a mathematical response. DMS includes dynamic geometry software (DGS), computer algebra systems (CAS), graphing calculators, some small web-applets, and many more. DMS has also been referred to as Mathematical Action Technology (MAT) by Dick and Hollebrands (2011). The products are too numerous to mention all by name, but large well-known examples include Desmos, GeoGebra, Geometer's Sketchpad, LogoWriter, and Mathematica. In each of these cases, the object of the software is to model a concept and provide the user interactive access to that model. Some DMS products, such as GeoGebra and Desmos Activity Builder, are themselves tools to create and distribute applets that could themselves be DMS (or AMS). Even a simple calculator is a basic example of DMS modelling addition: the user inputs 2+3 via the buttons and receives 5 as feedback.

AMS has different roles for learner and teacher. An AMS "calculator" may ask the user the sum of 2+3, then compare the input for accuracy against '5'. A learner is offered exercises or problems to respond to, while a teacher is offered the data, report, or analysis of their student's responses. Again there are numerous products, but well known examples are Cognitive Tutor and Khan Academy. AMS includes software also called Intelligent Tutoring Systems "ITS" or

Adaptive Learning “AL.” AMS inherently asks a question of the user and requires an input—an answer to assess—before providing feedback. An AMS ‘calculator’ would ask the user to input the sum of 2+3, compare the input to a stored answer key, then keep track if the input matched the stored answer.

The specific DMS products considered in this study are GeoGebra and Desmos, since they are freely distributed online and thus highly accessible. GeoGebra refers to itself as Dynamic Mathematics Software while Desmos describes itself as a graphing calculator and activity platform (Desmos, 2016a; GeoGebra, 2016). In each, a user interacts with objects on the screen using the mouse, a touchscreen, or by writing instructions in text or code. In this manner, users are able to interact with “concrete” representations of mathematical objects in ways that may be physically difficult or impossible (Sarama & Clements, 2009). Scrubbing[1] through parameters of a quadratic function or classically constructing the circumcenter of a triangle and then changing the triangle offer the user continuous feedback in the form of the constantly updating graph or diagram. These “concrete” visual representations share some of the role of real-world manipulatives in the classroom. But their digital nature allows real time alterations, transformations, and interactions that have ramifications for the learning and pedagogy involved in their use.

A *digital manipulative* is an DMS object with certain mathematical properties. These properties are intentionally designed and programmed so that they are discoverable by the student. Just as with a physical manipulative, the student can ideally intuit the properties of the manipulative as transferable and generalizable to the mathematical concept at large. What

[1] “Scrubbing” in the context of film/audio production refers to moving back and forth along the film or recording’s timeline in the attempt of finding a particular time location. For DMS: moving the value of a parameter back and forth on a slider in the attempt to find a particular result.

separates digital manipulatives from their physical counterparts is the vast expansion of the possibilities for manipulation. Whereas the compass and straightedge have been geometric tools for thousands of years, it is only on the computer that their marks and constructions may be edited post facto or continuously. A triangle constructed with DMS is no longer a singular example, instead it is representative of an infinite set of triangles defined by the properties enforced by the method of construction, a set which may be explored with simple clicks and drags of the mouse (Clements & McMillen, 1996).This power is not limited to Geometry, as algebraic equations and parameters can also be explored and handled by analogue (M. Hohenwarter & Fuchs, 2004).

This study is a qualitative case study of the implementation of lessons built on DMS and AMS. I sought to answer the following research question: how do DMS- and AMS-based lessons differ in supporting student learning? The main lenses will be student discourse, and type of engagement with the software. The subject classroom is an Algebra 2 class for 10^{th}-12^{th} grade students at a high school. The class is a socioeconomically and ethnically diverse set of students in public schools in the San Francisco Bay Area. The teacher has three years of experience, all at this school. All teachers are former colleagues of mine and/or I have served in part as their instructional coach, both formally and informally. As the teachers implement the lesson I will study how students and teachers talk about the mathematical concepts of the lesson with Desmos and GeoGebra as the lesson's materials, how students react to and interpret the feedback given by the programs, and how teachers use what they observe in their own students to shape and guide the lesson. In this fashion I will reveal some of the properties of AMS and DMS centered lessons, the affordances they offer, and the considerations needed for better curriculum design around them.

Chapter 2: Literature Review

This study is based on research and theory that place DMS as an extension of physical manipulatives while also considering the unique capabilities enabled by computers. I also touch briefly on cognition theory about how students learn and come to know a mathematical concept. Finally, I review some specific papers that analyze cases of DMS and AMS implementations.

Concrete to Abstract

The basis for DMS is in the larger topic of manipulatives. Manipulatives are objects that serve to model a concept by embodying certain properties of that concept and allowing physical manipulation by a learner. For example, tracing-folding paper could be used to discuss geometric reflections, as a pre-image may be traced easily to find its reflected image. Manipulative use itself is based in the philosophy that abstract mathematical thought is built only upon and abstracted from "concrete" ideas with which the student has had prior experience (Piaget, 1952; van Hiele, 1986). Piaget ascribes cognitive development as beginning with sensorimotor experiences in the first two years of life; objects are learned to persist beyond the immediate senses of the present. Young children then conceive of a world made of concrete objects, but have difficulty with classifications and properties that extend past the immediate senses. From ages seven to eleven, the child gains the ability to use "operations" such as transferability and reversibility of properties. Finally, development through age twenty allows the child to conceive hypothetical and use abstract deduction (Piaget, 1952).

Similarly, van Hiele proposed a set of five sequential levels that classify how students learn Geometry: 0: visualization, 1: analysis of properties, 2: abstraction and transference of properties, 3: deduction, 4: rigorous systemization. Importantly, van Hiele argues that the progression is not automatic with age. Mayberry (1983) demonstrated that even adults who have completed theoretically rigorous geometry courses may present only at level 0 or 1. Van Hiele recommends that in order to attain rigor, learners must be properly experienced with all preceding levels, and that mathematics teaching should be focused on advancing students through those levels.

Should we learn about math in the manner that Piaget says we learn about the world? Amalgamating a set of experiences to make inductive inferences until such a time that we are capable of convincing ourselves via deduction? This philosophy is in line with Pickering's (1995) description of advances in new concepts in formal mathematics and science: new concepts are targeted (bridging), then based upon previous models and knowledge (transcription) then discrepancies are handled by modifying the model to fit both old and new (filling). Pickering also notes the concept of human agency in the actions of bridging and filling. And if agency is a tool used by all professional mathematicians and scientists, then it should be an element of the learning environment for students. Thus, Boaler and Greeno (2000) assert that didactic, procedure-based lessons engage only on well-trodden paths, yielding "received knower" students with an under-developed intuition and an over-dependence on their instructors. A learning episode in which the students themselves exercise the agency of bridging and filling is one that Boaler and Greeno hypothesize yields fewer received-knowledge knowers. Opportunities for students to discuss their ideas and form arguments can be facilitated in part via DMS and digital manipulatives.

Harel (2013) bases his theory of intellectual need in Piaget's ideas as well. In Harel's Duality, Necessity, and Repeated reasoning (DNR) conceptual framework, the knowledge of mathematics is composed of the sum ways of understanding and ways of thinking in a community. "Different individuals are likely to produce different ways of understanding […], students engaged in a dynamic geometry software activity may carry out conjecturing and justifying acts and, accordingly, produce different conjectures and justifications." (Harel, 2013, p. 121) Knowing is a resolution of tension between assimilating and accommodating ideas that turns disequilibrium into equilibrium in Piaget's model. In other words, Piaget says the only way to know something is when the knowing balances the mental concept with the environment concept. Assimilating and accommodating is like Pickering's actions of bridging, transcribing, and filling. Perturbations to the equilibrium mean old knowledge cannot be directly transcribed or assimilated and must be filled in or accommodated with new knowledge. Harel argues that for students to learn mathematics, they must have these perturbations that create a need for the concept. The need is not an "affective need"— such as obligation to satisfy the teacher or social/cultural pressures—but an "intellectual need" such as to seek equilibrium between the environment and the mind's concept of that environment (Piaget, 1985). Harel's theory says that any piece of knowledge is connected to a problematic situation from which that knowledge arose. The problematic situations are subjective to the person or community. Harel defines five categories of intellectual need: need for certainty, logical causality, computation, communication, and structure. Harel's Necessity Principle, "For students to learn what we intend to teach them, they must have [an intellectual] need for it," has been adopted by Desmos as a guiding principle in their design (Luberoff, 2015).

DMS as a Manipulative

With DMS, a problematic situation can potentially be synthesized in ways beyond what is possible with the spoken or written word. Sarama and Clements (2009) outline seven digital affordances (DAs) that DMS may offer:

DA1. Bringing Mathematical Ideas and Processes to Conscious Awareness

DA2. Encouraging and Facilitating Complete, Precise Explanations

DA3. Supporting Mental "Actions on Objects"

DA4. Changing the Very Nature of the Manipulative

DA5. Symbolizing Mathematical Concepts

DA6. Linking the Concrete and the Symbolic with Feedback

DA7. Recording and Replaying Students' Actions

DA1 and DA5 seem particularly relevant towards synthesizing Harel's problematic situation for a student to access. A student must be aware of an idea in order to feel any intellectual need related to that idea.

Previous studies of lessons based on use of manipulatives (technological or physical) have demonstrated positive results for manipulatives used in a long-term (yearlong) setting when compared to abstract symbolic based lessons (Sowell, 1989). Other studies caution that manipulatives themselves are not "magic;" the manipulatives are not inherently carriers of mathematical concepts (Ball, 1992; Sarama & Clements, 2009). Ball notes that there is a common assumption that students who use manipulatives will draw correct conclusions: "students may well see and do other things with the materials." Ball, at that time, hypothesizes that rules about "proper" use of the manipulatives can alleviate some of the issues. But does that merely replace a written algorithm with a physical one? Consider Ball's example of children using bundles of sticks as manipulatives for place value in the context of subtraction:

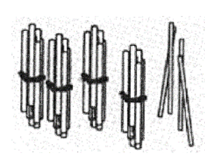

Figure 1: "44 – 27"

Although the manipulatives in this case are relatively flexible, teachers will usually tell students that they must always group by tens and that when they need to subtract, they cannot do it unless they unbundle an entire group of ten. Without such instructions, many second graders I know would simply remove a few sticks from a bundle—just enough to make the subtraction possible. But instead they follow the rules. This works very well: students unbundle a group of ten and count that they have fourteen sticks. Next they take away seven sticks. They then take two bundles of ten sticks away from the remaining three bundles and they happily write down 17 (Ball, 1992, p18). Ball is concerned with manipulative efficacy since "students sometimes learn to use manipulatives only in a rote manner. They perform the correct steps but have learned little more." (Clements & McMillen, 1996) Here, Ball notes that students who happily followed the procedure of the manipulative then still struggled with the "training wheels removed." Clements & McMillen agree, "actual base-ten blocks can be so clumsy and the manipulations so disconnected one from the other that students may see only the trees—manipulations of many pieces—and miss the forest—place-value ideas."

What Ball perhaps did not consider at the time was the possibility that while the manipulative itself was a good concrete starting point for learning, it was weakened by a reliance on authoritative and arbitrary rules from the teacher. The rules of the game were arbitrary because they do not link to inherent properties of the object. The bundles are not inherently tens, and tens are not inherently bundles. The strategies learned by the students do not extend beyond the sticks, because the properties of the sticks also require the teacher's rule set in order to be abstracted. The students had little agency in the vignette of subtraction, and had no opportunity to perform bridging or filling (in fact, their attempts to bridge were actively denied by teacher rules). The rules were authoritative in the sense that students following the rules could be merely falling into a learning identity in which students feel their obligation is to follow directions rather than understand the mathematics.

Normative identity is defined as “an aspect of the classroom social structure, concerns the role of an effective mathematics student as it is constituted in a particular classroom.” The students interpret mathematical norms in the context of social norms, such that a student’s understanding of a mathematical concept is influenced by the social structures in the classroom (Cobb, Gresalfi, & Hodge, 2009; Yackel & Cobb, 1996). In the bundles of ten activity, students who were socially obligated to follow teacher direction would perform certain actions with the bundles. However, these actions did not align well with strategies that were applied beyond the setting of the bundles. The obligation was to satisfy the narrow teacher direction leaving little room for exploration. Clements & McMillen (1996) offer that computers change the way in which teachers can enforce mathematical norms, “the computer blocks can be more manageable and ‘clean’.” Technological manipulatives such as DMS offer teachers flexibility and customization that is lacking in physical counterparts. Consider a possible computerized version of the bundles of ten manipulative: separating single sticks from a bundle causes unbundling, and having ten or more loose sticks causes a bundle to be formed. In this manner, the student’s exploration is not bound by an obligation to vocalized rules set by the teacher. The rule still exists, but it is now an internalized property of the manipulative. The norm of bundling tens is cleanly and consistently established by the behavior of the applet: it is impossible to not do it. The technology transforms the bundling norm from a rule that must be followed to a property of the environment. Consider that we do not need to obey gravity, it merely exists. But in addition, the technology would allow the teacher to remove this scaffold later (turning that property off) and discuss with students what the norm was useful for. The customization of the computerized tool allows teachers to directly scaffold mathematical norms by first automating them, then directly discussing them in abstraction. Cobb, Gresalfi, and Hodge found that when the goals of a lesson are broader—exploration instead of mechanization—then students view their obligations as sensible and reasonable. The student's "vehement" discussions led the researchers to suggest the students internalized their obligations. Students were "coming to view themselves as having the ability, facility and legitimacy" to contribute and be responsible for the mathematics in their classroom.

DMS representations of objects are mathematically durable and scalable. A constructed object on the screen—such as a rhombus—may be altered by changing components without affecting its defined representation. For example, if the angle between two sides is changed, the object still retains properties sufficient for it to remain a rhombus. A physical model perhaps made with straws and string would deform unless serious precautions were observed. Other DMS constructions use sophisticated combinations of capabilities to provide a manipulative that would be difficult or impossible physically. For example, the graph of a parabola can be adjusted continuously by its parameters to fit an image of projectile motion.

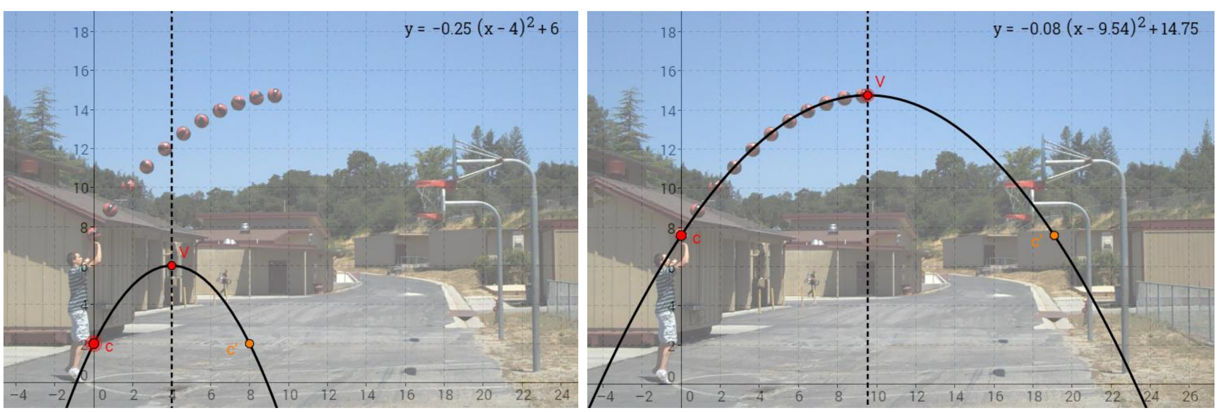

Figure 2 Parameters are scrubbed by dragging point c (bound to the y-axis) and V (unbound) in GeoGebra. *http://www.geogebratube.org/student/b87258#material/87254* *adapted from* *http://blog.mrmeyer.com/2010/wcydwt-will-it-hit-the-hoop/*

In this manner, applicable mathematical norms may be enforced non-authoritatively: a non-rhombus is impossible not because it is against the teacher's instructions, but because it is against the programmed capabilities of the given DMS applet.

Furthermore, in a computer environment, additional scaffolding is possible that blurs the lines between physical representations and symbolic representations of numbers. Ball's past view of manipulatives might consider them only applicable at van Hiele level 0 or 1. A manipulative that abstracts itself in parallel to a developing student's knowledge offers new teaching modalities to research.

Interaction modes

LogoWriter (also known as Turtle Graphics) is one of the early computerized environments used in classrooms. Students provide instructions that guide an animated turtle around the screen, and the students must formalize their language so that it can be interpreted by the computer (Figure 3). In a study with fourth graders, Clements & Battista, (1990), had strong results: "by the third interview all of the Logo children possessed a mathematically accurate concept of angle." These students interacted with mathematical concepts via the tools of programming. A student must write Instructions such as RT 90 to turn the turtle to the right, 90 degrees. Clements' & Battista's data shows the Logo group offering more sophisticated definitions of angles, while the comparison group held onto the notion that an angle's size was determined by the length of its sides. The students were held to blunt standards by the computer. "TURN RIGHT" is not an accepted instruction, and it may be that the norms enforced by the computer are interpreted in a different manner than one enforced by a human. Note, these blunt standards are notably around instructions for the software's

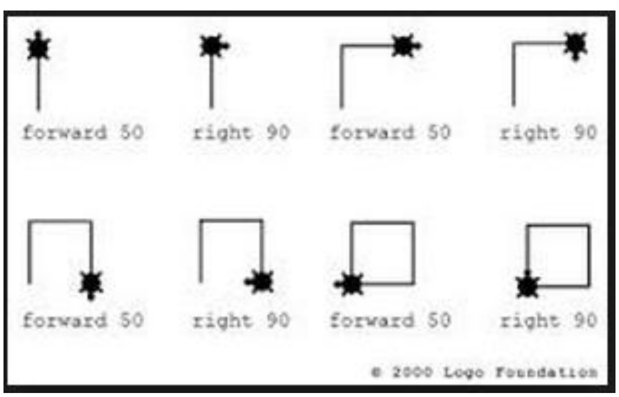

Figure 3 Example of eight sequential LogoWriter commands with their results.

operation not about a learning objective, therefore LogoWriter remains DMS not AMS. “Students are not surprised that the computer does not understand natural language and that they must formalize their ideas to communicate them (Clements & McMillen, 1996). Students formalize about five times more often using computers than they do using paper (Hoyles, Healy, & Sutherland, 1991).

The mouse offers entirely different input modalities (J. Hohenwarter, Hohenwarter, & Lavicza, 2009). Arzarello et. al. (2002) describe “dragging” modes in which a user moves the location of a point, possibly affecting dependent objects within the DMS environment. Subcategories include “wandering” and “guided” dragging used when students were discovering properties in low van Hiele levels, for example: “dummy locus” during phases when students were formulating conjectures, and “dragging test” as students tested their conjectures. The researchers here noted that students initially did not take advantage of any type of dragging, but that once the tool was used, students “[shifted] from working in Cabri to theoretical reasoning. In some instances, the students anticipate a conjecture and then they check it in Cabri, while in other cases they ‘read’ what is happening in Cabri and make conjectures from this.” The students in this study were making transitions from van Hiele levels 1 and 2 up to 2 and 3. Or, Pickering might describe “dummy locus” dragging as a method of filling: the student is attempting to drag a point along a locus known only to him- or herself in their own conjecture.

Clements and McMillen would agree, “Computer environments may be unique in furnishing problem-solving scaffolding that allows students to build on their initial intuitive visual approaches and construct more analytic approaches. In this way, early concepts and strategies may be precursors of more sophisticated mathematics.”(1996). Kysh (1999) describes this scaffolding taking place as part of a cyclical learning path: students latch language and ideas onto their initial experimentation which aids in forming their beginning concepts.

Designers of such programs note the versatility of concept presentation, but also note capabilities not fully understood yet. Moyer, Bolyard, & Spikell (2002) posited, “along with new ways of manipulating visual representations, entirely new forms of representation may emerge, including manipulatives that have no counterparts in the physical realm,” perhaps predicting capabilities exemplified in DMS now. It is a new development that allows mathematical analysis of a quadratic model to be “scrubbed”—changed by physical motions of the mouse—via geometric analogues (Hohenwarter & Fuchs, 2004; Sträßer, 2002).

DMS in Relation to Classroom Discourse

If “speech unites the cognitive and social” (Cazden, 2001), it is unclear with whom the “dragging” student is speaking and socializing. Echoing Ball—manipulatives are not magic—“physical and computer manipulatives can be useful, but they will be more so when used in comprehensive, well-planned, instructional settings” (Sarama & Clements, 2009). It is not enough for a manipulative to be present, even one capable of scaffolding at all van Hiele levels. Instead, it must be brought to life by the skilled teacher. Cazden notes that students gain much of their learning from linguistic structures, so that the intended curriculum is modulated by the mode of communication.

A student’s interpretation of a concept may be based upon pattern recognition of the discourse. In the setting of a triadic dialogue (teacher initiates a question, draws a brief response, then evaluates with a typically narrow focus) the student is facing a normative identity pressure to conform to the teacher’s ideas about mathematics or produce a response that is socially over mathematically acceptable (Cazden, 2001; Yackel & Cobb, 1996; Zevenbergen, 2000). These

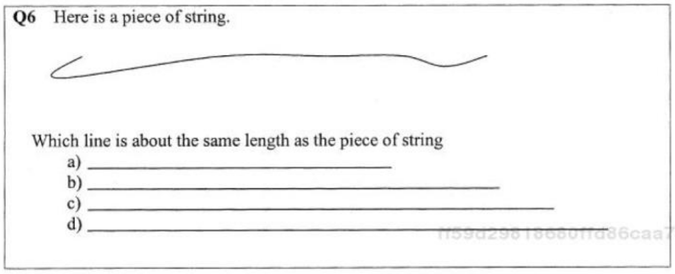

Figure 4 A non-lingual task. 1997 Queensland Year 6 Test

language-based structures are barriers to learning, a code that must be cracked before learning takes place. Language itself may also be a barrier, but visuals can help reduce or circumvent the barrier. Zevenbergen found indigenous and nonindigenous students in Australia performed equally well on a test item in which the task could be visually assumed without reading the text (Figure 4). The results suggested "similar mathematical competence among the students, not accessed through the other [non-visual] test items" (2000, p. 208).

To establish a classroom discourse that allows students to learn, the students must be able to talk about their own ideas and experiences with mathematics. "The opportunity for learning is influenced by what students are helped to coproduce through dialogue" (Walshaw & Anthony, 2008). DMS implementation may fit well into Lampert's (1990) definition of the teacher's role in the class. Lampert says teachers need to do something other than explain to the students why the rules work; students learn how truth gets established "as they zig-zag between their own observations and generalization." Those two modes occupy van Hiele levels 1 and 2, but importantly lay the groundwork for 3 and 4. So if a teacher uses a virtual manipulative as a guide for mathematical norms, then the students form their ideas about normative identity from the consistency only a computer may bring, while the teacher's role is decentralized: serving as an ally and scaffolder towards understanding rather than the dispenser. Boaler and Greeno, along with Walshaw & Anthony (2008) and Hand (2010) put forth the flexible decentralized classroom as one in which students have the opportunities to explore concepts via bridging and filling and discussion with the peers, which in turn allows for multiple ways of being competent in mathematics, not merely fitting into a strict mono-cultured code.

Contrasting Assessment Math Software with Dynamic Math Software

To contrast the features of DMS, this field study will also look at Khan Academy's missions and exercises as an example of AMS. The goals of AMS are centered around determining if a student's responses are correct or not, and potentially offering support materials such as hints or a video lesson. In the 1990s and early 2000s, Cognitive Tutor was developed at Carnegie Mellon University to support learning of secondary mathematics. The use of the software has been associated with higher achievement scores, with claims that the software is "more effective than classroom instruction and normal human one-on-one tutors but not yet as effective as the best human tutors." (Aleven & Koedinger, 2002) But Aleven and Koedinger also acknowledge the issue with shallow learning, "students were better at finding answers to test problems than they were at articulating the reasons that were presumably involved in finding these answers." In their 2002 study, they found that geometry students who are further required by the AMS to justify responses with the proper reasoning (supplied by selecting from a glossary—Figure 15) out-perform students only required to provide an answer. They still caution that explanation "by reference" does not stop all shallow reasoning. The distinguishing features of this AMS Geometry Cognitive Tutor compared to DMS is that its feedback is independent of the meaning of the students' input. The input is only compared to the software's internal key. The feedback to the student then is limited to telling the student if their input was right (a match) or wrong (anything not matching). Different AMS software has differing capabilities: some will accept $52 - 22$ if the input should be 30, while others may not accept $1/2$ if the answer should be 0.5.

Khan Academy (KA) began as a resource of videos on YouTube (Khan, 2012). Khan began tutoring a young relative in 2004 on mathematics concepts, solving example problems and giving some exposition to the underlying concept. Since 2010, the focus of KA has shifted from the video tutorials themselves, towards guided series of online exercises. And in 2013, KA redesigned the structure of the skills from a single large dependency tree to a K-12 grade-level-aligned list (Murphy, Gallagher, Krumm, Mislevy, & Hafter, 2014). Currently, within the domain of Mathematics, users of KA progress through a set of skills by correctly answering questions. KA keeps data on the accuracy of the responses, whether or not a hint was used, and the amount of time spent on each question. As student working within a skill must also achieve certain goals such as "Get 3 correct in a row" (Figure 5). From this data, KA will decide if the user either needs more practice, or has mastered the skill. Summaries of the user's data can be seen by the user herself, or as part of a report to the user's teacher or parent. To motivate users to continue working, missions reward "energy points" and "achievement badges" that can be displayed on the user's profile. This type of motivation is sometimes called gamification since it an extrinsic reward system modeled after what is found in video games.

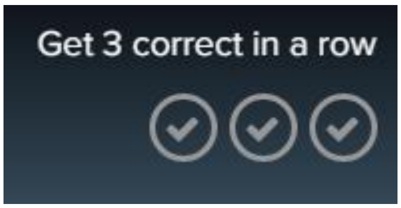

Figure 5 a KA interface element

While the thousands of videos are still available on YouTube without the exercise system, KA now describes itself as "playing a fundamental role in reimagining education in a way that empowers teachers and students to build personalized, mastery-based, and interactive learning environments that foster creativity and collaboration." (Khan & Slavitt, 2013) KA's exercise sets and videos are available as supplements for any teacher. Teachers may use school computers and class instruction time for students to work on KA, or teachers may employ a "flipped classroom" style, in which KA videos or exercises are assigned for students to complete outside of class. The videos alone are not AMS, they are merely information; the exercise sets give feedback (sometimes linking to a relevant video) and are therefore the core AMS element of KA.

The goals of AMS like Khan Academy place them at a different stage of learning that that occupied by DMS. Elizabeth Slavitt, KA's then Math Content Lead, says, "for us, our goal isn't necessarily that Khan introduces new concepts to students. We want to give practice" (Madda, 2014). KA sees the advent of the internet, and the current wave of technology opening new avenues to re-think how education is conceived from the ground up. "What will make [successful education reform] attainable is the enlightened use of technology… The idea is to integrate the technology into how we teach and learn," Khan explains in his book (2012). KA's goals are centered around personalized learning, perhaps meaning the software determines the appropriate topic and exercise difficulty level for the student. This sounds on the surface like the software's goal is to find a zone of proximal development (Vygotsky, 1978) between the exercises that individual student can do and those they cannot do. But the software does not offer and remove scaffolding: the hints and videos are always available and not specifically tailored to the student's inputs.

As AMS goals are usually different, KA does not closely use any of Sarama and Clements "digital affordances." However, Puentedura proposes a model called SAMR (Substitution, Augmentation, Modification, Redefinition) for thinking about and classifying technology in education. He illustrates the categories with the example of a word processor. At first, word processors were substitutes for typewriters, allowing much the same functionality. But with additions of functions such as backspace or cut and paste, the technology is augmenting the writing process. When the word processor gains functionality of integrating with spreadsheets and graphic packages, the task of writing is significantly modified, and further integration such as multiple users on a single document or writing for the web with linked items and the writing process is redefined (Puentedura, 2006). The categories are not mutually exclusive for any one piece of software, as a word processor occupies all of the spaces at once. The SAMR model perhaps more closely applies to the implementation and use rather than the technology itself. Puentedura argues that feedback should be more than right/wrong. In agreement with Wiliam, he argues that scores or praise alone do not “move learners forward” but comments do. Providing scores at all can be counterproductive to moving learning (Puentedura, 2014; Wiliam, 2002).

Implementation Studies

Empirical studies of the use of DMS and AMS typically focus on learning outcomes. In one study, kindergartners working with DMS based shape-patterns created a larger number of patterns, and used more elements in their patterns compared to a group using physical manipulatives (Moyer, Niezgoda, & Stanley, 2005). There is evidence that instant feedback afforded by computers is perceived as “natural consequences” by learners (Thompson, 1992). Thompson also notes that restrictions of possible inputs may lead to “more meaningful” use of symbols, if the constraints are reflections of mathematical concepts. Akin to Clements’ noticing the “blunt standards” of a computer input box can improve student communication. A study of pre-service teachers in the UK saw gains in visual spatial skills (Baki, Kosa, & Guven, 2011) and students who manipulated an object indirectly (via altering property or a parameter) in DMS performed better on assessments (Sedig, 2008).

Specific attention has been given to ‘dragging’ as an interaction mode distinct from non-computer lessons (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Wanko, Edwards, & Phelps, 2012). In a DGS environment a user can directly move an object, which may then also indirectly move or alter another object on the screen following a pre-designed rule either implemented by the software, the lesson designer, the teacher or the user. Baccaglini-Frank & Mariotti aimed their study at conjecture generation based on the indirect motion in DGS reflecting logical dependency chains (2010, p. 229). Their observations include that "students use the words 'try it' with respect to an idea (or possibly [a] yet unexpressed conjecture) when they intend to perform a robust dragging test. [students drag a point to see if properties hold for an "arbitrary" construction with their steps]" (2010, p. 249). Finally, they conclude that dragging in DGS may contribute to the "conjecturing phase, giving the solver a new means for developing arguments."

Khan Academy asserts that they do not envision being a replacement for a teacher, but instead, a resource (Madda, 2014). In 2013, KA redesigned the structure of their skills from a large dependency tree spanning all levels to a K-12 grade-aligned list In the SRI foundation's large report, "Khan Academy played the greatest role in supporting their instruction by providing students with practice opportunities (82%)" (Murphy et al., 2014). In the pilot study, the school sites were given free reign over the usage of KA, and most found it useful, but not revolutionary, "KA use represented less than 10% of schedule math instructional time at [all but one (22%)] of the pilot sites." Furthermore, time spent on Khan Academy dropped at 16 out of the 18 sites that used KA in the first school year. Finally, teachers had little use for the assessment reports generated by the exercise sets, 59% of SY2012-2013 teachers checked KA reports "once a month or less or not at all". However, those teachers that checked reports regularly (one or more times per week) found the reports "useful."

As DMS and AMS grow in usage, researchers have found new pedagogical and curricular ideas to be aware of. Unless the lesson is designed well, students using DMS may more easily grasp the “forwards” direction of a geometric proof at the expense of the backwards “only if” direction (Lachmy & Koichu, 2014). Task design can play an important role in students’ acquisition of a concept. Johnson argues that incorporating dynamic segments along the axes of a graph in addition to the (x, y) point itself helped draw students’ attention to two changing variables instead of just one. Other features to attend to include the purposeful inclusion or exclusion of numerical measurements, altering the graphical representations (and using many) and being mindful of the potentially complicating use of time as a dimension in lessons around co-variation (Johnson, 2015, 2016; Johnson, McClintock, & Hornbein, 2016).

Teacher as a Facilitator

Teacher training for the use of AMS or DMS is lacking, meaning teachers are often left to themselves to learn how to facilitate DMS-discourse based lessons. For over twenty years, there have been goals in mind to shift the teacher’s role to that of a facilitator (Hannafin, Burruss, & Little, 2001). But often the resistance comes from the teachers’ discomfort with relinquishing control of the classroom (Hannafin et al., 2001) One teacher in Hannafin’s study noted that despite her students working hard and making powerful connections, she also had to face student misconceptions in a more direct and frequent manner: they were no longer hidden by the student coasting through lecture. The teacher in question also felt pressure for the students to perform on an achievement test. Stols and Kriek (2011) of South Africa contend that another important variable is the teacher’s general proficiency with computers. Other studies (Dove & Hollenbrands, 2013; Moyer, 2001) note that the teacher’s computer experience may show up in the variety of types of scaffolds they use in their attempts to integrate virtual manipulatives into their lessons. Dove & Hollenbrands further noted that teacher created files may not always be faithful to the generalization of a concept. Researchers in Turkey also emphasize that technologically sophisticated teachers are necessary for advancement of technologically sophisticated learning environments (Erbas, Cakiroglu, Aydin, & Beser, 2006). While developers themselves hold widespread training (J. Hohenwarter et al., 2009) classrooms around the world are not uniformly ready to shift to a virtual manipulative mode, despite strong interest.

Where my study fits in

While technology is becoming more widely adopted there is not complete agreement on which technologies are useful for which purposes in education. In addition, the principles of task design, curriculum design and pedagogy are not fully explored. DMS and AMS offer very different capabilities but are lumped together in public perception as “edTech.” This case study of a secondary class using both DMS and AMS can serve to illustrate the qualities of use for both types and connect the students’ and teacher’s experiences to theory. In the study, I investigate some of the design intentions and results from students using Desmos and Khan Academy.

Chapter 3: Methodology

This qualitative study investigates one teacher's use of two existing technological curricula: Desmos Activities, and Khan Academy. How do DMS- and AMS-based lessons differ in supporting student learning? The main lenses will be student discourse, and type of engagement with the software. The lessons took place in an Algebra II course during a unit on Trigonometry. The teacher planned his unit with input from me on the capabilities of Desmos. The teacher had independently chosen to also use Khan Academy. I observed three consecutive days of the students participating in the lessons. Through the observations and data collection I will position the use-cases in the pedagogical theory around how students are learning through these software programs and how the teacher uses the software.

Selection of Participants

In Summer 2015, I reached out to former colleagues and teachers I had coached who were currently implementing or interested in using mathematics software in their teaching. Four teachers from three school sites around the San Francisco Bay Area responded and we set a rough timeline for me to observe during times when they planned to use computers. I observed teachers who were available early in the year in order to develop the observation protocol and select the specific software to be used. These preliminary observations took place in middle school classrooms around the Bay Area, along with a high school Algebra 2 classroom. For the formal observations, I chose the high school class, taught by Mr. Drake[2], since he already planned to use both Desmos and Khan Academy in his classroom in early Spring 2016. Mr. Drake was a first year teacher at my school in the East Bay Area in 2013-2014 and is now in his third full year of teaching. He teaches Algebra II and Statistics.

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[2] pseudonym

Setting

Mr. Drake's class is an Algebra II course made up of 10th, 11th and 12th grade students. The school is a large socio-economically diverse comprehensive high school in the East San Francisco Bay Area. Data for the 2014-2015 school year show, 79.8% of students were socioeconomically disadvantaged and 17% were English Language Learners. The enrollment was 39.5% Hispanic or Latino, 33.2% Black or African American, 15.6% Asian, 6.6% White, and no other group larger than 1.4%. Finally, 15% of 11th grade students, who were tested, met or exceeded the state standards for mathematics in 2014-2015, compared to 23% district-wide, and 33% state-wide. The ethnic descriptions of those that met or exceeded state standards in math was 18% Hispanic or Latino, 4% Black or African American, 70% Asian, and 6% white (California Department of Education, 2015).

Mr. Drake signs his class up for access to a class set of laptops in order to do software-based lessons. He selected to use Desmos Activities and Khan Academy due to the free cost, online availability and teacher structure. When we first talked early in the school year, he considered GeoGebra as well but he ultimately did not use it as the basis for an entire day's lesson during the observation window. Mr. Drake decided on using two students per laptop for Desmos and one student per laptop for Khan Academy.

Instruments

The primary instrument was field notes, taken during the observed lessons. These consisted of free-form handwritten notes along with a structured observation protocol. In addition, I collected student responses that were accessible through the teacher portals of both Desmos and Khan Academy. Finally, I recorded a time-lapse video of the overlaid student responses to the first graph of the Function Carnival Activity (Desmos, 2016b), in the manner of teacher David Cox (2014).

My researcher notes are supplemented by audio and video recordings in the classroom. In this manner I gathered student dialogue as they worked. The recordings are from digital audio recorders on student desks, on my person, and on Mr. Drake's person, and a video recorder focused on the room as a whole. While taking notes, I marked what I considered noteworthy student talk with a rough time index in order to locate it in the audio recordings later. This allowed details of student dialogue to be filled in from the recordings.

The function of the structured observation protocol was to observe key features of student engagement with the computers based on a rubric (Appendix A). The template for the rubric was developed by the Ubiquitous Computing Implementation Research project (UCIR, 2006). I adapted the rubric with trial runs and expert input from Dr. Judith Kysh of SFSU. In practice, I was only able to focus on one or two categories at a time, I chose "Student Talk" and "Student reaction to indication of error."

Both Desmos and Khan Academy gather student inputs and provide that information in a teacher view. Desmos records the drawings and typed inputs of the students. I saved copies of the class's overlaid aggregated responses along with a selection of individual student inputs after the lessons were over. Khan Academy provides the teacher summary reports of the students' progress. I saved a copy of the aggregate report along with a selection of individual student reports when the lessons were over.

Procedures

In the informal observations of Fall 2015, I observed Ms. Allen and Ms. Beale's classes participate in Desmos Activity "Central Park." I took notes on student dialogue and debriefed with the teachers afterwards, discussing the lesson. From these observations I identified aspects of the student interaction with the software which I used to prepare for future observations. I selected the UCIR observation protocol as a basis for structure then tested it in a trial run in Mr. Drake's classroom as the students worked on Khan Academy. I then reviewed the template and observation protocol with Dr. Judith Kysh of SFSU for feasibility and validity.

The formal observations took place in Late February 2016 in Mr. Drake's class. Mr. Drake decided to use "Function Carnival" (Desmos, 2016b) activity for this lesson (Figure 6).

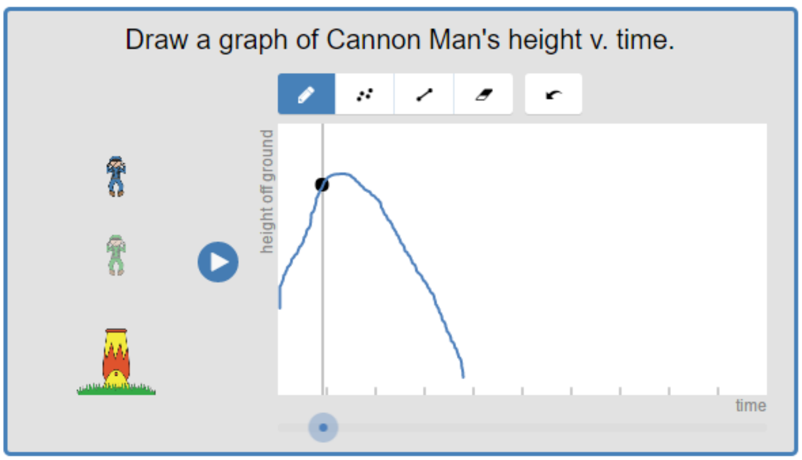


Figure 6 The student's interface for Cannon Man as part of Function Carnival. The blue man represents any drawn y-value at the marked time, the green man shows the actual y-value at the same time.

The observation period was chosen to be 3rd period to accommodate last minute lesson plan changes after Mr. Drake ran the same lesson in 1st period and had a prep period 2nd. During his 2nd period I placed the audio and video recorders. During the observation, I circulated the room and made handwritten notes along with using the structured observation protocol. My own interactions with students were limited to responding to quick questions about the logistics of getting to or continuing the lesson. From the start of the lesson until Mr. Drake moved students past the first graph in Function Carnival, the Desmos teacher's view of the aggregate graph was screen-capture recorded using Mr. Drake's laptop.

During the next day, students worked on Khan Academy as part of Mr. Drake's participation in Khan Academy's Learn Storm 2016[3]. Students spend each Tuesday in class— and an optional amount of time outside of class— doing Khan Academy lessons as suggested by Mr. Drake. This observation took place on the third Tuesday of Kahn Academy usage. As with the Desmos Activity I observed by circulating throughout the room while the audio and video recording devices recorded. There is no special screen capture recording for this day. I again restricted my interactions with students to providing responses to requests for logistical help in starting or continuing the lesson. At the end of the period, Mr. Drake asked how many people watched "at least one Khan video" and if they "found it helpful"; I recorded these counts and student comments.

After each lesson, recordings were collected from each recording device and the student response data was saved from Mr. Drake's teacher interface, and I aligned audio, video, and input recordings with my notes to match student comments and select episodes of interest. When a particular episode was selected, I transcribed relevant audio making anonymous the identities of the students.

As Mr. Drake and I were talking after the lesson, another math teacher at the school walked in and was interested in doing the lesson herself. This teacher, Ms. Edwards, is a veteran teacher at the school and has also worked as department chair and district professional development. We quickly transferred the laptops to her room next-door and she ran the lesson for her 5th period Algebra 2 class. With Ms. Edwards permission, Mr. Drake and I observed informally and also screen-captured recorded the teacher view.

[3] http://www.learnstorm2016.org/bay-area/overview

Data Analysis

I imported all audio and video files into Adobe Premiere Pro CC 2015 then used the audio matching function to synchronize all the tracks. I played the merged tracks once through and wrote small snippets of audible dialogue along with the timestamp. Using these timestamps and my observation notes as a guide, I scrubbed to specific points in the merged recordings to attempt to transcribe student conversations or describe notable elements of the aggregated graphs. I would mute and unmute tracks ad-hoc, and use the video to watch for gestures or my location in the room (with my microphone) to transcribe more accurately. Finally, I created a zoomed-in timelapse clip of the teacher's aggregate view of the Cannon Man graph, condensing 10 minutes of classtime to 26 seconds. I created a similar time-lapse for Ms. Edwards' 5th period's Cannon Man graphs.

Using the categories on the structured observation, categories from the literature, and other emergent categories, I coded the selected episode dialogue for common features and attempted to correspond the groups dialogue to their contemporaneous computer input. The coding aims at connecting student engagement with DMS and AMS to qualities in existing theory.

The analysis is not comprehensive towards capturing all student dialogue. Example student dialogue cannot be assumed to be representative. No quantitative comparison can be made between dialogue from the Desmos lesson versus dialogue from the Khan Academy lesson. However, the dialogue does indicate existence of different qualities during each day.

The screen captures of the Desmos lesson are also coded for representing student actions and representing student ideas. Mr. Drake took still screenshots of each group's final product which I then analyzed for tool usage and assessed on a basic rubric for precision (Tables 1 and 2).

Validity

I have an existing relationship with each of the teachers involved in this study. I am either a current or former coach of their teaching practice. In each case I assisted or facilitated the planning or implementation of the lessons using math software. Thus these classrooms are cases of teachers already interested and willing to implement DMS and AMS lessons and who further received direct support in doing so. However, no teacher is specially trained. Mr. Cortez and Mr. Drake had not implemented computer-based lessons before this year. The perspective for this study is that of a teacher-coach who supports the implementation of DMS or AMS in secondary math classrooms.

Between 2007 and 2014 in teaching my own classes, I implemented DMS activities using Geometer's Sketchpad and GeoGebra with increasing frequency. I implemented AMS activities through Cognitive Tutor in 2007-2009 but did not use any AMS after 2009. I have also advocated for the use of DMS style lessons at education conferences (Farrar, 2013). I attempted to minimize researcher bias by selecting classrooms already planning to use the software and by triangulating data with teacher reflections vs. my field notes vs. records of the software.

Ethical Considerations

This study's proposal was reviewed by the SFSU Office of Research and Sponsored Programs and declared exempt. The procedures and observations are within the bounds of normal teaching and coaching practice. Each participating teacher volunteered. To protect confidentiality, student identities are coded. Furthermore, the actual names and other identifiers of teachers, schools, and districts are not used. Audio and video recordings will be deleted and erased after completion of this report.

Chapter 4: Results

During the DMS lesson, the software supported specific learning via its designed communication of mathematical norms. There was more discourse around justifications in the DMS day while more discourse around mathematical procedure and definitions in the AMS day. The students had many more opportunities for interaction with the computer with DMS compared to AMS.

During the Desmos lesson, students worked in pairs to create graphs matching the animated situation. The pairs refined their graphs over time and used varying strategies. They also responded to the written prompts in 1-4 sentences or phrases, sometimes returning to the previous graph to review or test information. Mr. Drake paused the work to have a whole class discussion three times, during which he would show parts of the Desmos Teacher View on the projector and ask specific groups to explain their reasoning. This structured the time to approximately ten minutes on Cannon Man, a two-minute discussion, twelve minutes on Bumper Cars, a six-minute discussion, six minutes on Ferris Wheel, and finally a two-minute discussion ended by the bell. The Khan Academy day was the next day. Mr. Drake had students work individually on KA in their personal accounts for the final 40 minutes of the period[4].

Desmos Day

Mr. Drake gave no guidelines for the graphs but did prime the students with a warm-up question, "Think of a time you saw a real-world graph and what it told you." This question was rephrased from 1st period as those students tried to respond with definitions to, "what is a real

[4] The warm-up activity for the second day was done via Kahoot (https://getkahoot.com/), a web-based "answer clicker" AMS in which students respond to multiple choice questions via their phones or a computer after entering a class code. The questions are timed and a histogram of the selections is shown immediately following the timer's expiration, and students earn "points" in the game based on their speed and if their answer matches the key. After each question, Mr. Drake would then demonstrate the solution. Mr. Drake gave four questions on converting between radian and degree angle measures. This made the second day's class period almost all AMS based.

world graph and what does it tell you?" Mr. Drake used this as the only introduction to the activity: seeding the idea that graphs convey information about "real world" phenomena.

During the period, Mr. Drake developed an emphasis on a particular property of each graphing situation based upon the class' work. For Cannon Man, he focused on the piecewise joint between the quadratic and linear sections of the fall: where the man opens his parachute. For Bumper Cars, he focused again on the piecewise joint but particularly the section after the joint: when the car had crashed and did not move. The Ferris Wheel discussion was cut short due to time but Mr. Drake had plans to note the symmetric properties of the sine wave and the interpretation of the midline.

Desmos Graphing

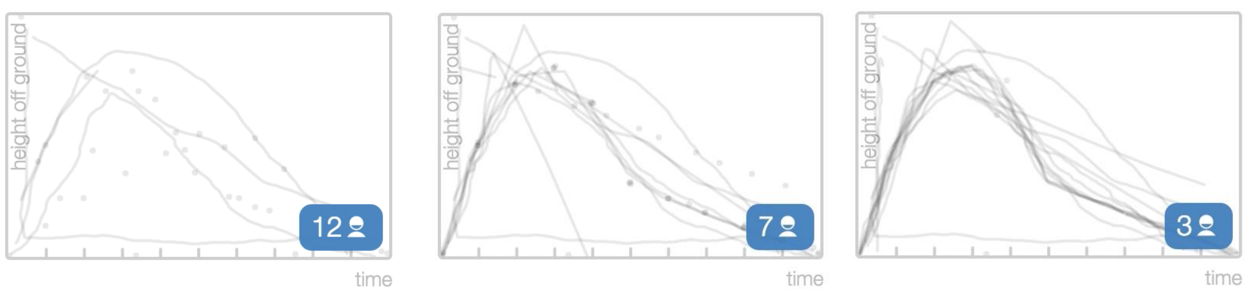


Figure 7 Teacher view of all Cannon Man responses after three, six, and ten minutes. The number indicates count of pairs currently on that screen.

In the first graphing slide, Cannon Man, most groups of students made their graphs more precise over time (Figure 7). This could indicate an internalized obligation to precision encouraged by the feedback of the animation. All groups begin by drawing roughly with either the freehand pencil tool[5] or adding individual points with the point tool. Twelve out of thirteen pairs[6] had drawn a rough graph and played back the resulting animation within one to two

[5] The drawing tools available to the students were: Freehand Pencil, Point, Line Segment, Eraser, Undo, and Clear All

[6] One pair logged in only to have their laptop battery die. They started again with a different name, showing up as a 14th pair in some raw data.

minutes of beginning the activity (Group 6 had drawn a right triangle freehand, immediately moved to the next slide and were then unengaged until ten minutes later.) Students appeared to find the feature of the playback of their graph useful as one student describes, “oh you can test with the little… pencil thing… and move it around, and the blue guy moves with *your* little guy—so you can see what things would look like.”

Despite no introduction to the tools, the students used a variety of graphing methods. The freehand tool (the pencil) was used most often, both in observed working practice and as evident in the final product. The point tool was used by groups during working practice but showed up in only two groups’ final products, as the Desmos software will remove any points that a freehand or line segment passes through. Some groups used the markings on the x-axis as a guide for precise graphing. A student in a whole class discussion, “we pretty much just played the video and then stopped it at the you know where the little notches are… the intervals […] and then wherever it was during that we pretty much put a point there.” Students used each of the eraser, undo, and clear-all tools frequently, often opting to start over completely. The line segment tool was evident in about half of the groups final products. All but two groups’ final products were internally consistent for tool usage; those who used line segments continued to use line segments. Group 10 did not draw through their points for both Bumper Cars and Ferris Wheel. Group 11 switched to the freehand tool for Ferris Wheel after using line segments previously.

Students experimented with graph behavior before submitting their final products. This shows agency in their learning. The time lapse video (Farrar, 2016) of Cannon Man illustrates many revisions and erasures. Especially earlier within each allotted time for graphing, groups would attempt nonsensical non-matching graphs or seemingly random points and drawings, but these would be quickly erased. In Ms. Edward’s class some students interpreted the goal of the

Bumper Cars slide was to steer the car safely through the course using the graph as a steering mechanism. One student was eager to show his off to the class (Figure 8).

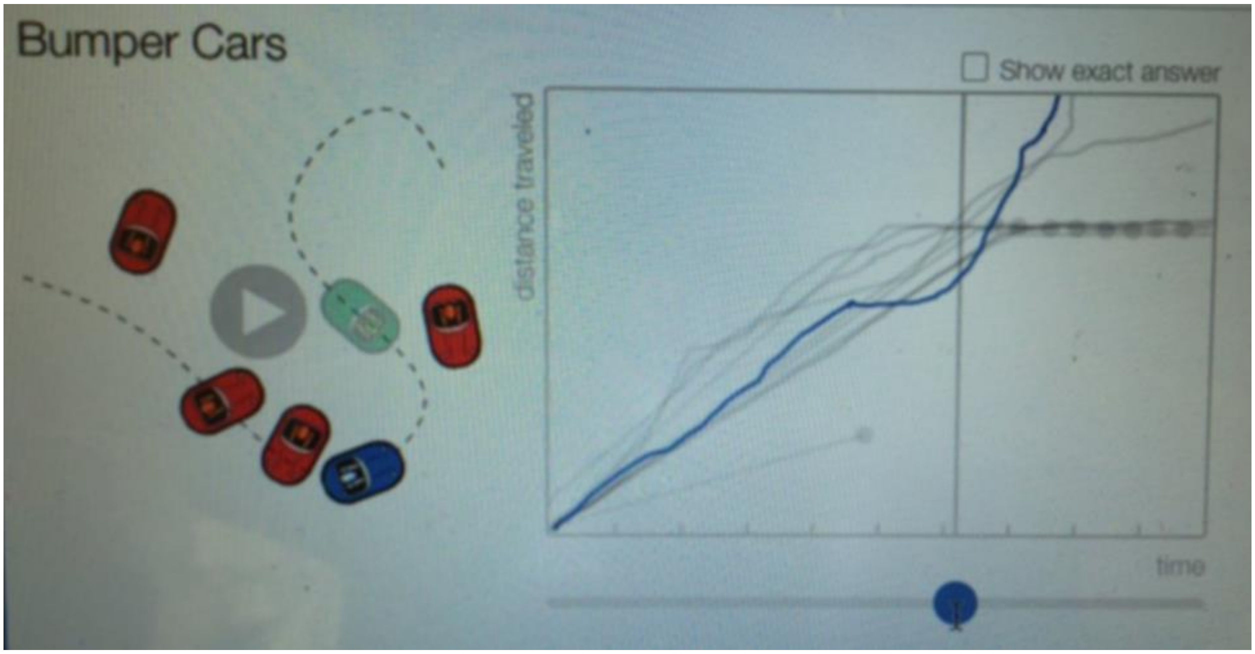


Figure 8 A student in Ms. Edwards class steers the blue car safely through the red cars using the distance v. time graph

In the screencapture video of Mr. Drake's class for Cannon Man, there is a general trend towards a tighter fit over time. But the tightness of the fit is not uniform along the x-axis. The quadratic-fall section of the graph appears more tightly contained than the linear fall (parachute) section. Eight groups did not feature a discernible transition between the parabolic and linear pieces. There was a similar split shown in the video of Ms. Edwards' class responses: some groups showed evidence they noticed the piecewise transition and others did not.

Often the "wrong" graphs led to student inquiry when the results were played back in the animation. The following exchange happened as Group 12 worked on the Ferris Wheels slide:

> S1: why are there four?
> [T notices the group's graph and comes over to their table]
> T: can I see what you guys drew?
> S1 oh no please haha I just drew
> S2 it looks like a horse
> T: it does look like a horse. What happened?

S1 it put like four [Ferris wheel carts]
S2: there were four going at the same time
S2: [playing the graph again] look look that line goes like… haha

The surprise indicates a perturbation to the students existing mental model Upon hearing this conversation recording one month later, Mr. Drake said he liked that the task had many possible "takeaways" and that he could draw out ideas about "things like input/output or multiple outputs" as they came up. Back on the activity day, Mr. Drake continued prompting the group to interpret the results of their graph:

T: ok stop there, why are there three different—
S1: cause the lines… the lines aren't all attached together
T: what do you mean?
S1: cause the lines are like… distributed in different places
T: [attempting to understand her meaning] distributed in different places…
S1: I don't know I just wanted to sound smart
T: no no, go on
S1: so they're not attached
T: oh so there's like different lines
S1: yeah yeah they're not attached
T: ok so what would different lines mean in this graph?
S1: there's like one for each [referring to the carts and the lines, but unclear in which priority]
T: ok so how many do we want?
S1, S2: just one!

The students may not have fully reached an equilibrium with the multiple cars represented and a definition of function as having only one output for each input. However, the design of the program utilized DA1 and DA7 (replaying student actions and bringing an abstract concept to the forefront) to bend this mathematical rule: the students are allowed to draw a *relation* in a problem that requires *functions*. This rule is bent but other rules are made impossible to break: such as scaling the axes appropriately and drawing only in the first quadrant. Those two items may be a roadblock for some students that do not allow access to the intended problem: the software's design, however, lets all the students access the intended activity by making those mathematical norms impossible to break. Recall the bundles-of-ten manipulative ‘s proper use required obeying authoritative rules: but in Function Carnival, drawing outside the area is impossible, and all drawings inside are interpreted (Ball, 1992; Boaler & Greeno, 2000; Harel, 2013; Piaget, 1985; Sarama & Clements, 2009; Yackel & Cobb, 1996).

Desmos Graph Interpretation

After each graphing situation, Desmos shows a sample student graph (Figure 9) with a conceptual error. The prompt asks, “Help Eric. What does his graph say will happen to Cannon Man? What would you say to help him fix it?” Students are given a text input box with a submit button, the typed responses are continually updated to the teacher’s view but are otherwise unprocessed by Desmos. Each graph contains one or more errors that Desmos expects some students to make.

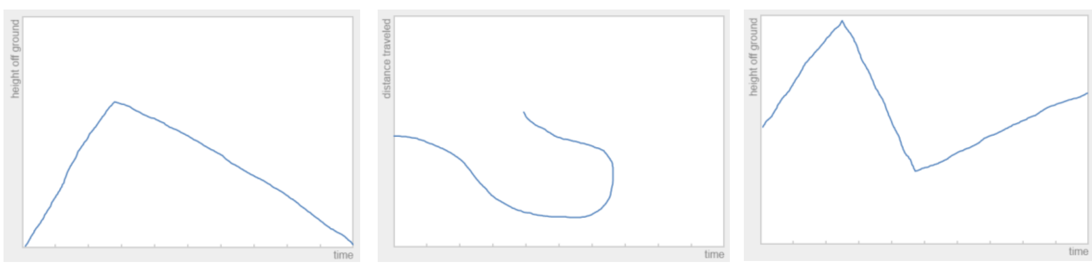


Figure 9 Desmos-supplied erroneous graphs for students to interpret. Cannon Man, Bumper Cars, Ferris Wheel

In Cannon Man, the sample graph very imprecisely describes the animation but does not show any indication of the change of speed with the parachute opening, nor a change from parabola to linear function. In Bumper Cars, the sample graph traces the directional path of the car, instead of distance traveled. In Ferris Wheel, the sample graph is linear and irregular instead of sinusoidal. Students who tried graphs similar to the given sample may recognize the errors and describe them, students who had accurate graphs may realize the errors without having tried them. Students who had inaccurate graphs that did not match the given samples may be unsure of what to write, although the software allows them to go back and try it directly.

Mr. Drake said—in reviewing the responses one month later— his students “wrote more than what [he] expected.” He guessed that on a hypothetical paper assignment asking similar questions he would have gotten shorter, perhaps one-word, responses. He also felt the students were more comfortable (compared to a hypothetical paper assignment) in writing responses that were not perfect, letting him capture more of their thinking. During the class period, Mr. Drake checked in on the teacher view of the class responses a handful of times, using it as a way to know which groups to talk with and which groups to call on in the whole-class discussion. The focus properties of the graphs were present in the writing of eight groups for Cannon Man, eight groups for Bumper Cars, and six groups for Ferris Wheel. All groups had at least one written response that referred to an emphasized property.

Students were sensitive to how the graphs portrayed and suggested speed. Despite the written response screens being non-interactive, 16 out of 35 responses used words relating to the speed of the portrayed situation— quicker, speed, gradually, constant, pace, slowly, fast, rapidly, rate— or the “speed” of the graph itself: Group 7 wrote, “the graph moves at a smoother and regular [pace.]” The word “constant” was also used with a mixture of meanings. Group 5 either misuses the word or is potentially referring to gravity, “As the cannon blows him up, he will have a constant speed going downwards.” But Group 11 describes linear as constant: “your graph [the provided erroneous graph] shows a constant move of him […]” In Group 10’s Bumper Car response it is unclear how they have interpreted the provided graph but the use constant to mean ‘unchanging’ path or speed irrespective of the axes: “Jenny’s graph states a constant path […] She would need to fix the constant of the distance over the time of the car crash.”

Desmos Class Discussions

Mr. Drake used the whole class discussions to share and formalize student ideas. He used the teacher interface to pre-select groups he wanted to call upon, in addition to taking volunteers. The following exchange is the entirety of the discussion that took place between the students' time on Cannon Man and Bumper Cars. Students are indicated by their group, or S if they could not be identified in this discussion:

T: most of you put this little-- I dunno-- can someone talk about this right here?
S: that dent?
T: yeah [waits] oh, did you want to talk about it?
Ss: [murmurs]
S: that's when he pulled the parachute
T: ok what happens when he pulls the parachute?
Ss: slowed down
G12: he hit a constant
T: ok I'm following, but how does this [linear part of graph] represent him slowing down?
G2: cause he was going fast and then, and then he kinda -- he doesn't stop but like it all goes in like slow down
T: slow down from what?
G2: going really fast
G7: its his distance back to the ground… its taking him longer to get back down to the ground
T: so its taking him longer to get closer to the ground
G7: yeah
T: which is why we're seeing the decrease. Ok, cool. Alright move on to #2

In this discussion we see students attempting to connect features of the graph with their existing knowledge about the context of a parachuted fall. The students in Groups 12, 2 and 7 are at different places but each has reasoned something about why the graph has its particular shape and features. The Cannon Man discussion lasted approximately three minutes. Mr. Drake wished in retrospect he could have assessed if more students understood the portrayed dimensions like the student from group 7.

Mr. Drake saw the discussions as vehicles to talk about the dimensions being represented. "If they could get a good understanding of distance… or height off ground […] and through the discussion its easier to grasp that… then they'd have an easier time moving forward with a graph representing situations." In the post-interview that he felt the discussion helped bring out student interpretations of the graphs: "students are talking about speed, but speed isn't represented explicitly." He further noted the student mentioning "constant" and guessed that she was referring to the rate at which he's falling (after the parachute opens). Later in the period, during the discussion on Bumper Cars, a student refers back to Cannon Man but appears to confuse constant rate with constant distance:

T: how do you know? How does this [the horizontal segment after the car crash] represent that your distance is traveling?
S: cause its flat
T: ok but why does flat represent distance on—
S: cause if you point it down, that's basically like, the little cannon man: he's slowing down. But the car isn't slowing down, he stopped. It made a sudden stop because he crashed.
T: what would happen if it was going up?
S: the distance gets farther but like on a straight line the distance is the same.

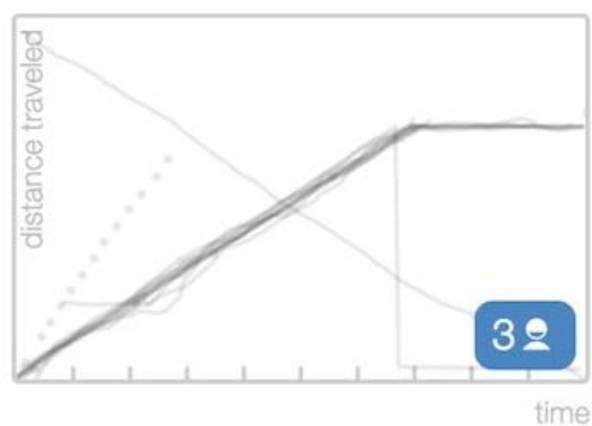


Figure 10 Aggregate T graph view at time of the Bumper Car discussion

Mr. Drake displayed the aggregate view of all the class's graphs (Figure 10) to begin the discussion on Cannon Man and Bumper Cars. He played back selected group's graphs with everyone able to see the corresponding animation for that highlighted group. He also displayed student written responses and had other students comment on them. Upon displaying Group 2's Bumper Car written response of, "Jenny followed the trail instead of tracing the distance the car has travelled. In her graph you can see that the car goes backwards. This does not correlate with the distance travelled."

T: so the first thing I liked about this was: "followed the trail" of the bumper car. Why is that different than what we needed?
S: because we needed distance
S2: it wanted to show the distance over time
T: yeah it didn't want you to show like where it went… it wanted you to show how far it has travelled. And remember we had time [draws blank axes on board] and distance
T: so its showing here I've travelled in some direction but I'm always adding to my distance

Mr. Drake reflected on using the Desmos task for multiple purposes. He had intended the Ferris Wheel section to be the entry point into sinusoidal graphing, but he felt it was valuable to spend most of the time on the concept of graphical display of functions since students "appeared to have the need." He wondered about focusing on or being attentive to the students' different uses of "constant" in future lessons. He also noted the time constraints, wondering aloud momentarily if he would do one graph per day over three days next time.

Khan Academy Day

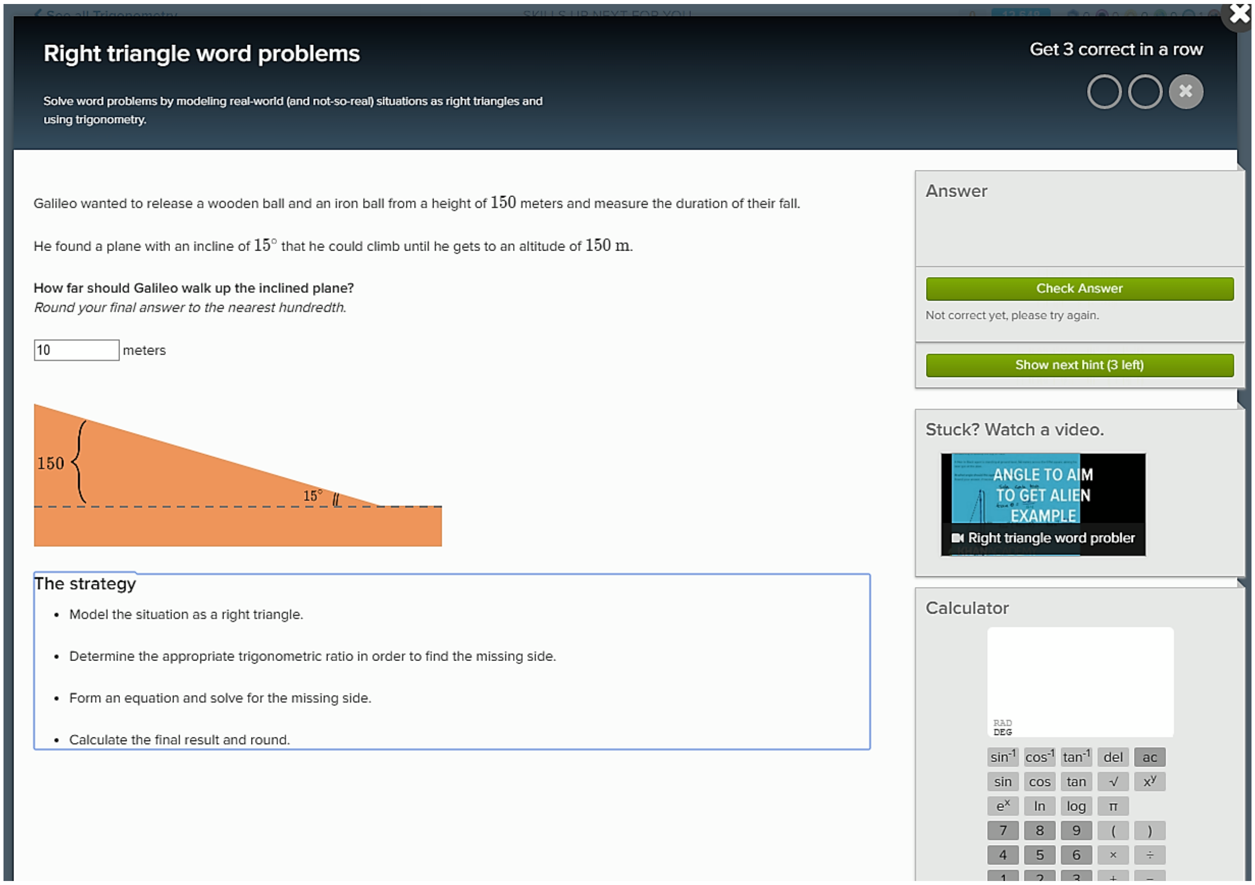


Figure 11 A sample student KA interface after submitting the incorrect answer "10" and clicking Show Hint

This was the third of five Tuesdays that Mr. Drake had his class work on Khan Academy in class. Mr. Drake gave the students a mission[7] to complete the Trigonometry unit. The students used approximately 38 minutes of class time logged into Khan Academy on this particular day.

Most student talk during the KA worktime was centered on procedures and definitions. The definitions of the trigonometric ratios were common, “sine is adjacent over hypotenuse”, “I thought it was opposite.” Second most common comments were about accuracy/grading/status. “I have 39% [progress]” “Oh I have 5%” Students showed apprehension before submitting an answer, aware that a mistake would mean more work. “I better get this right […] what the

[7] https://www.khanacademy.org/coach-res/k12-classrooms/best-practices-k12/a/what-is-a-mission

freak!? I always get it wrong." I observed no cases of students conjecturing or reasoning beyond the problem at hand. Cases of students justifying their statements were mostly typified by referring to given definitions of the trigonometric ratios.

When the KA software indicated an error, some students were initially frustrated, but then attempted a variety of next steps. Some would catch their error quickly, the student who said he "always get[s] it wrong" immediately followed up with "oh I always forget the π." Most often students would turn to a peer for help. In this episode, one girl is helping peers on either side of her, they are working on finding missing sides in a right triangle with trigonometry equations.

S2: you forgot 50… after the sine
S3: so its 2… divided by… sin 50 ?
S2: x
S3: x?
S2: this [points] is multiply
S3: so its 2 divided by sin 50 x
S2: [hesitates] oh… you skipped one step. oh ok those numbers don't divide by x you cross multiply. Erase this part… keep the 2. So 2 times [inaudible] by itself right? Equals sin 50. x. [unclear if she means sin(50)/x or sin(50)x or sin(50x) or something else]

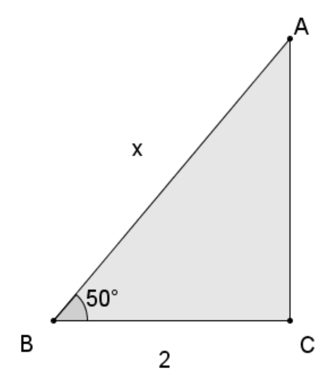
Students preferred the peer interaction over the videos: twelve out of 25 students reported watching "at least one" video during the day, seven "found it helpful." I observed most students involved in intermittent peer discussions throughout the period: three students chose to sit alone. Mr. Drake reflected that the students often sought a peer who had just completed a question in the skill. No students moved seats during this period, so this would be limited to the tables of four at which the students were already sitting. S3 in this episode is still working on the problem of finding a missing side, and has been marked incorrect by KA at least once. She has not watched the video and it is unclear if she asked for a hint, but she did ask S2 for help again. They work with the onscreen calculator but are also writing via KA's provided scratchpad using the trackpad. S3 is writing there:

S3: so divide by x… you get an x on both sides
[S1 calls S2's attention for a moment]
S3: wait does x go inside the 50? where's x on the calculator?
S2: there is no x.
S1: so what's the equation for sine?
S2: sine is adjacent over hypotenuse
S1: I thought it was opposite
S2: use cosiiiine! No its truuue
S1: you're confusing me. Look, this is the hypotenuse. Use the hypotenuse and the angle to find the opposite side. That's sine.
S2: … then .. I did it wrong [grabs her HW from last night]
S3: sine aint working

These conversations are procedural to the point of attempting to instruct how to write an equation. The definitions are confused and while KA can provide them the definitions those hints and videos are not utilized by these students.

Students also frequently asked for help from Mr. Drake; he said most of his time was spent helping students complete problems if their peers could not completely help. In this episode typical of the period, he comes up to the table as the students continue. S3 is still working on the same problem (see figure below)

S2: I'm s'posed to use cosine I've used the sine the whole time!
S1: cosine is adjacent over hypotenuse
S3: yah, sine is opposite over hypotenuse
S2: this is why I hate math. It just confuses me
[T walks over]
T: you just broke my heart
S1: its too confusing. When to use sines and cos and tan
[T sits with S3]
T: ok let's start with what we know. What do you know about this triangle?
S3: umm you have to find the x
T: ok and what. What are we trying to find? Which side is missing?
S3: opposite?
T: ok good.
S3: [points] isn't this the adjacent?
T: yep so we know the adjacent, and what's the adjacent?
S3: [inaudible]



T: ok cool but when you say adjacent we need to know adjacent to what. Because, if I start here [points] then this side is adjacent. So what angle do we want?

S3: B.

T: B! and how much is angle B?

[T and S3 continue for another minute with a procedural solution of the problem]

Mr. Drake began with questions assessing the student's knowledge of the problem. She initially interprets the "missing side" as the side without a measurement, and this belief is not directly addressed. KA is inert during this exchange.

Mr. Drake said he could not directly rely on KA's reports of student achievement. In reflection, he said students would perform well on some skills but then struggle on other basic ones. He had assigned students the Trigonometry mission but he knew his students needed differentiation, "Khan tries to account for this with the fundamental problems, [a diagnostic set before the mission] but I think my students got bogged down in that: students were always behind, and then put further behind." He said KA assumed that students in the Trigonometry lesson would "know Geometry," but that he felt was not a reasonable assumption.

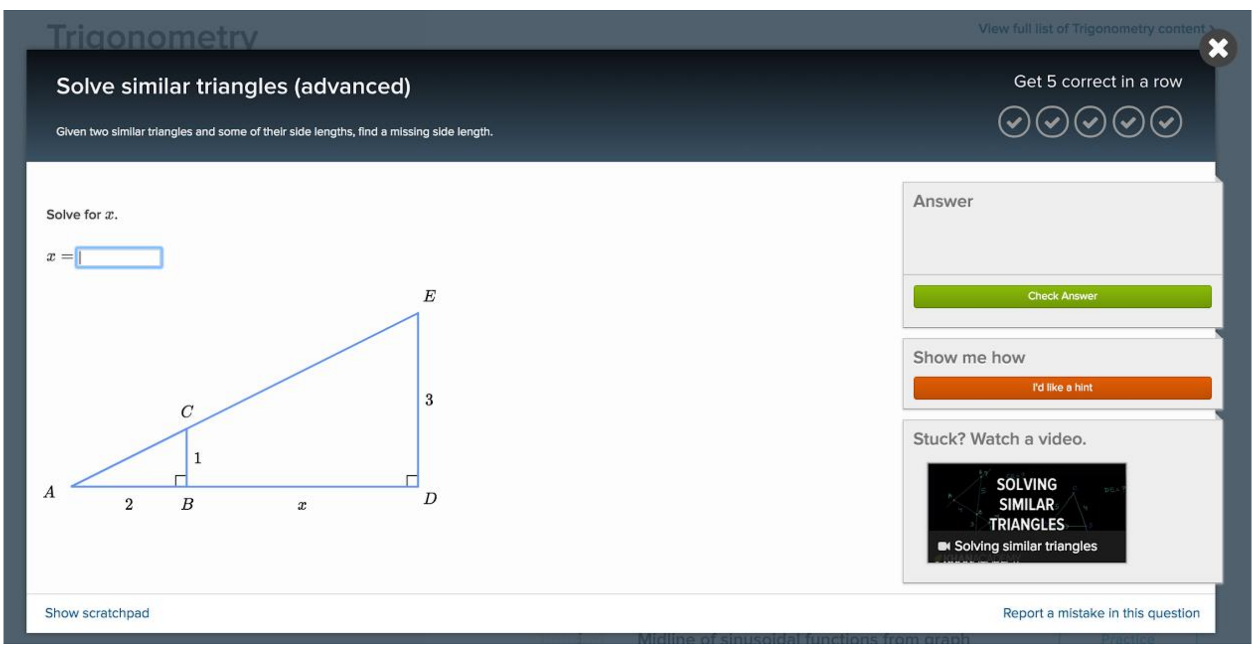


Figure 12 A KA question in the Fundamentals for the Trigonometry Mission

He pointed out a particular “fundamentals” question he felt his class would struggle with (Figure 12) “Vocabulary like transversals or alternate interior in the first 30 sec of KA’s explanation video would immediately throw them off.”

Reflection with Mr. Drake One Month Later

Mr. Drake later used another Desmos activity: Marbleslides with his Algebra 2 class. He also continued using Khan Academy one day per week in class during his trigonometry unit (two additional weeks for a total of five weeks). He had expressed interest in future usage of Desmos lessons but felt that he might use a guiding worksheet to “layer on vocabulary” or plan to explicitly have a following lesson to formalize ideas from the explorations. He said he might build a lesson around screenshots of his students’ work available to him in the teacher view.

Voluntary use of KA outside of the regular Tuesday class session was low. The twenty-five student KA accounts were logged in for a median of 5 (mean 4.68) times Tuesdays between February 1st and April 15th. During the same time period, the same accounts logged in for a median of 1 day (mean 2.92) other than Tuesdays. Sixteen accounts logged in once or zero times outside of Tuesdays. Mr. Drake hypothesized that for his students either external supervision or extrinsic rewards would be needed to increase usage.

Mr. Drake listened to the audio recordings and reviewed the graphical and written responses. He described the “outcomes” of the Desmos activity as “increased academic discussion” even if the “language was “not as mathematical as we would like.” He described the “outcomes” of the KA days as “students who accomplished a task or skill felt accomplished” and that it served to prepare for similar interactions on the upcoming SBAC [Smarter Balanced Assessment Consortium, the state test for California]. When asked if anything surprised him about the lessons, he said the students talked and wrote more than he would have predicted during the Desmos lesson. “I was a little surprised by amount of [student to student] dialogue and [whole-class] discussion. I don’t think I would have gotten that writing on a worksheet.”

Mr. Drake said he would aim to use both Desmos and KA again in the future. He expressed the desire to sharpen the focus of KA’s mission “down to the skill level perhaps.” While he wanted to increase the time spent on a Desmos activity while also scaffolding in more definitions and “connecting it better to my unit.”

Comparison

The type of talk during the Khan Academy day is in contrast to that during the Desmos day, which had no procedural or definition talk. When the idea of definition of a function arose due to the graphs students drew, no student specifically mentioned it. There was some assessment talk during Desmos, students would get excited when their Cannon Man animation closely matched the original, “ooh damn near hella close! Yeah!” but this was *self*-assessment based on interpreting the feedback of Desmos’ animation. Students also got excited when KA told them their answer was correct.

There were no instances of KA’s exercises using any of Sarama and Clements (2009) digital affordances. Each problem was presented in a manner no different than it would be done on paper. The AMS within KA does evaluate the answers for accuracy so the primary difference between the KA exercises and the same exercises on paper is that a student cannot move on until they are correct in KA. In contrast, Desmos’s DMS directly brought the concept of functions to conscious awareness (DA1), linked concrete animations and symbolic graphing with feedback (DA6), recorded and replayed student actions (DA7), also turning a function’s graph into a manipulative itself (DA4). Neither KA or Desmos focused on symbolizing concepts or performing mental “actions” on objects in these particular lessons, but they might be present in other products from each company.

Desmos leaves much to the teacher, while KA can theoretically be done without teacher help. The presence of other students and the teacher while doing KA may have reduced the engagement with the hints and resources provided by KA. Desmos’s Function Carnival has no definitions or procedures—in fact, very little language at all. The teacher is an important element in how the concepts get developed after the opportunity is supported by the software. In the topic of overlap, the sine function appears as a result of a spinning Ferris Wheel in Desmos, while the sine function appears as a defined operation to perform in the KA exercises. The connections between these two ideas did not come up in either lesson.

Chapter 5: Discussion

In Mr. Drake's classroom, I see the DMS and AMS influencing the discourse in inherently different directions. The AMS Khan Academy set extrinsic incentives for teacher and students to talk procedurally and 'get answers.' The DMS Desmos' Function Carnival set few or no extrinsic incentives. By channeling information to the teacher and reflecting mathematical feedback to the students, the program set up the basis for potential conceptual discussions for the teacher to take advantage of. The National Council of Teachers of Mathematics also believes, "mathematical action technology [DMS] influences not only *how* teachers teach but also *what* they are able to teach" (Leinwand, Brahier, Huinker, & NCTM, 2014). AMS's support of individual practice hit bumps in the road in Mr. Drake's implementation, but might be useful in other capacities.

What software brings to the classroom

In Puentedura's SAMR (Substitution, Augmentation, Modification, Redefinition) model for discussing technology implementation, Khan Academy's exercise sets are mainly substituting for pencil and paper tasks—using questions and problems that work equally well without the computer. KA provides hints and videos, which substitute for teacher given hints or teacher example-based lectures. However, in Mr. Drake's class the students preferred to ask another student or the teacher to consulting the videos or hints. KA compiles data on student responses, which they might argue is augmentation in Puentedura's model. As students work, KA will decide whether to let a student move to a new skill or not via its algorithm of number of correct responses in a row; technically the software collects evidence and makes a dependent decision—the definition of formative assessment. But the data alone is not enough to evaluate student knowledge. The table working on trigonometric triangle lengths in KA had procedural and definitional struggles that KA did not specifically diagnose. It therefore did not provide Wiliam's (2002) "feedback that moves learning forward." In the SRI report (Murphy et al., 2014), teachers who viewed reports once per week found them useful, but Mr. Drake felt his students were too often "stuck" to make enough progress to generate useful data. The teachers studied by SRI were using KA from 11 to 90 minutes per week for the entire year, a fact that suggests there could be a threshold for amount of KA usage to generate meaningful data.

Desmos' Function Carnival is smaller scope in a way: a single lesson, replacing a similar pencil-and-paper incarnation such as Graphing Stories (Meyer, 2007). By utilizing Sarama and Clements digital affordances, the task takes on more qualities of Puentedura's SAMR model. Viewing the Cartesian plot of a function as "symbolic" and the animation as "concrete," Function Carnival's primary interaction links these two representations of an event with the feedback of replaying the consequences of the student's input. This weaves digital affordances 3, 6 and 7: mental "actions on objects," linking concrete and symbolic with feedback, and recording and replaying student actions. Function Carnival's drawing pad substitutes for the paper drawing. But student capabilities are augmented. Mr. Drake's students quickly and independently noticed new abilities: "Oh you can test… with the little pencil thing…", "we pretty much just played the video and then stopped it at the you know where the little notches are." Because Desmos augments the information by showing the situation or story in motion, the task becomes modified: students aim at improving their graph, attending to and interpreting analytic features of the situation. In the pencil-and-paper setting, a student may compare their graph to the "right answer" and move from there, but with Desmos's the use of the Digital Affordances allowed rapid interplay between students drawing, replaying, and feedback as visible in the aggregate view of the student graphs over the duration of the lesson. Puentedura says that technology allowing creation of new tasks that were previously inconceivable, means the technological implementation is a redefinition of the lesson. Ms. Edward's student began to steer his car to achieve a goal other than matching. He controlled the car by plotting a function, reminiscent of moving between rungs on Victor's Ladder of Abstraction (2011). Function Carnival provides a visual abstraction of situations, and Ms. Edward's student latched onto the abstraction to control the concrete. Function Carnival allows creation of new tasks along these lines, "stepping back down" in Victor's words, where the visual abstraction is the interaction mode.

Discourse: Student, Teacher and DMS

Function Carnival fit well in the discussion style lesson plan. As the mathematical norms such as distance v. time comparisons, Cartesian graphing in general, and definition of a function were instilled in the DMS, they formed the boundaries of a sandbox in which the students were free to explore. The burst of differing functions (and non-functions) graphed by Mr. Drake's and Ms. Edward's students gave way to a systemic progression towards precision. The students tested the boundaries of the medium, explored its capabilities and feedback then used that knowledge to create a meaningful response. The students display agency and creativity, which is enabled by the meaningful feedback of the software's interpretation of the graph (Boaler & Greeno, 2000; Cobb et al., 2009; Yackel & Cobb, 1996).

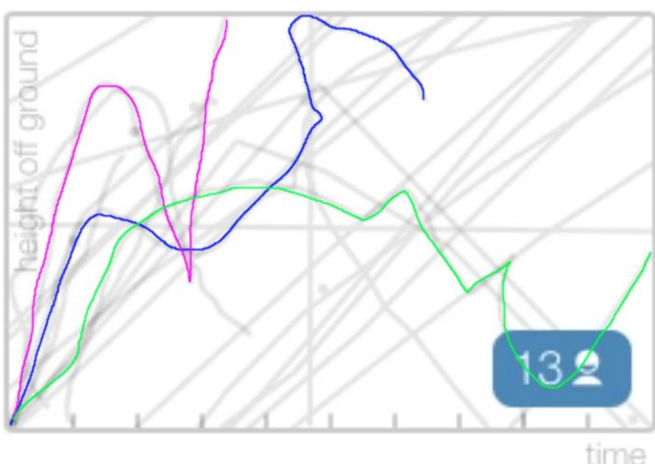


Figure 13 Three of Ms. Edward's groups responses (highlighted) in progress during Cannon Man. A fourth notable response is the scribble across the entire screen.

Students have the opportunity to zig-zag between observations and generalizations, as described by Lampert (1990). As they receive feedback from the software in the form of the animation playing out their graph, discrepancies from expected behavior are Piagetian perturbations. There was genuine shock in the voice of Mr. Drake's students when multiple copies of their Ferris Wheel car appeared in the play-back of a non-function, "why are the four!?" (Harel, 2013; Piaget, 1985). This comment is evidence of the students having the intellectual need described by Harel. A problematic situation has arisen, and the formal definition of function can serve to resolve that situation. The student first had agency and opportunity to investigate small, unvoiced conjectures such as 'what if I draw …' gaining empirical experience with the behavior of the function and its corresponding animation. Since the DMS feedback was meaningful (beyond right and wrong) the student was able to move forward in the concept and notice the perturbation when four Ferris Wheel cars appeared. This fits Harel's "need for causality," but it also opens the door to a "need for structure." The mathematical feedback of the animation in this fashion brought the mathematical concept of function *near* "conscious awareness" described by Sarama and Clements as one of the affordances the digital medium gives. The early visual display of the concept is precursor of the sophisticated mathematics to come later (Clements & McMillen, 1996; Sarama & Clements, 2009). The teacher still occupied a large role in the direction of the conversations, Group 12 was very close to proposing a definition of a function in her conversation with Mr. Drake. With more experience now having done the lesson once, Mr. Drake may expect conversations such as this to arise and be more ready to drive the students conceptual learning.

> S1: there's like one for each [referring to the carts and the lines, but unclear in which priority]
> T: ok so how many do we want?

When helping this group, instead of “how many do we want?” he could try to promote an abstracted or transferred version of the student’s statement to another context. Perhaps “how many ‘lines’ are there for your distance from home today?” Or to press the student for justification that there should be “just one line.”

The use of well designed DMS still needs careful teacher planning to capitalize on opportunites to build conceptual knowledge. Mr. Drake’s initial plan to focus on the Ferris Wheel (for purposes of connecting with his trigonometry unit) may have caused him to miss chances for student conceptual argumentation and discourse about the other two situations. He himself was aware of this, as he reflected later that he might spread the lesson out over 2-3 days to allow more time for discussion. His comment in reflection on possible “takeaways” from the lesson provided ideas on possible topics for deeper discussions: definition of function, properties of analytic functions like extrema or increasing/decreasing, contextualizing a sine graph. Further, the students’ confusion around the representation of speed and the various uses of the word “constant” could be explored in a manner that builds intuition around Calculus derivatives.

T: What happens when he pulls the parachute?
Ss: slowed down
S: He hit a constant

To build off these contrasting remarks, the teacher could ask, “We see the class’s graphs for Cannon Man’s height, what does a graph of his *speed* look like?”[8] Then the teacher could then have the class come to an agreement around the usage of terms like “constant” and “slowed down.”

In the whole-class discussion of Cannon Man, one student describes the situation entirely using rates, perhaps only thinking of the animation in that moment: “cause he was going fast and

[8] At the time of this writing, a Desmos style feedback graph for Cannon Man of speed over time is not currently offered by Desmos itself. Although it seems technologically feasible, there may be design challenges.

then, and then he kinda -- he doesn't stop but like it all goes in like slow down." Another student describes the graphical behavior well in terms of distance, "it's his distance back to the ground… its taking him longer to get back down to the ground." Mr. Drake chooses to let those two comments hang as he tells the class to move to #2 (probably in the interest of time). The students were not in disagreement, but they may not have been understanding how the graph represented the situation in the same way. This would have been an opportunity, using Horn's (2008) accountable argumentation norms, to reconcile and clarify the differing ideas of students through an extended whole class and through discussion to develop precision and depth in defining students' understanding of distance, rate and time relationships. The Cannon Man discussion was one of several situations that provided an opportunity for further student-teacher discourse.

In Walshaw and Anthony's conslusion regarding teacher's role in student discourse, "student outcomes are not *caused* by teaching practices, but they can at least be *occasioned* by those practices" (2008, p. 542). DMS provided extra tools with which to occasion discourse. Ms. Edwards needed to be ready for and open to her student's 'wrong' graph. She was able to position the student as successful for exploring the different goal of graphing to have the car not crash. There are multiple ways for students to be successful with DMS if teachers are ready for the anticipated and unanticipated responses.

To take advantage of the occasions created by DMS teachers may need to do a different kind of preparation for their lessons. Teachers need time to work with the software and each other to anticipate and discuss potential student responses, so when both forseen or unforseen misconceptions arise they are ready to facilitate productive discusions. Careful planning by the teacher to anticipate what kinds of comments are likely can improve the teacher's readiness to have conceptual discussions with students. When Mr. Drake intervened with the group who exclaimed, “why are there four?!” he had not anticipated the students’ interpretation of what to graph, so spent most of his time with them trying to understand that before getting to a question that could prompt their rethinking. In viewing the recordings, Mr. Drake immediately commented on the opportunity the situation offered. This is reminiscent of Ball’s (1992) warning, “manipulatives are not magic.” While DMS offers opportunities, careful preparation and attention to pedagogy is still an important part of the lesson.

Johnson (2015) also predicted potential confusions relating to reasoning with time as a variable, as students may interpret time as either conceptual, which is separate from an individual’s experience of passing time or non-conceptual, which is purely experiential. There was significant overloading of terms relating to speed by students—most notably “constant.” It is unclear if these confusions are related to potentially different interpretations of time, but they are at least a significant indication of varying levels of conceptual understanding of analytic functions. Recall the student relating the flat line in Bumper Cars back to the Cannon Man graph in discussion, “cause if you point [the function] down, that’s basically like, the little cannon man: he’s slowing down. But the car isn’t slowing down, he stopped. It made a sudden stop because he crashed.” Here the student describes a function “pointing down” (decreasing) as “slowing down” instead of the decrease in distance traveled[9]. However, with such readily available student attention to rates, an extension task that asked for plots of velocity for Cannon Man or Bumper Cars may be a way to support and extend the learning in this area.

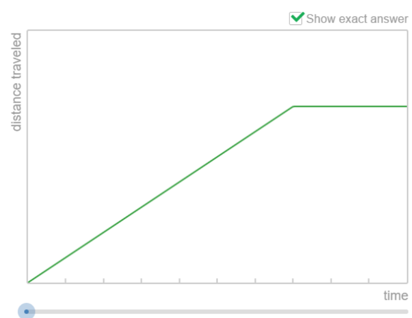


Figure 14 The graph that exactly matches the "Bumper Cars" animation

[9] A strict interpretation of “distance traveled” might not allow such a decrease at all. The label “distance from starting point” might be more accurate but also more verbose. The imprecision could be an opportunity for a class discussion and debate on the correct interpretation. Mr. Drake’s class did not discuss this directly, but Mr. Drake did once offer a truism offhand that “distance traveled” must always increase.

Function Carnival intentionally has no vocabulary or specific numbers. It becomes incumbent on the teacher to provide language opportunities and scaffolds through discourse as represented on the 2nd half of Kysh's (1999) cycle of learning. DMS most easily fits to promote the exploration and naïve conjecture portion of learning, but to begin to formalize and move to higher levels of understanding there needs to be supplementary efforts to move beyond DMS's initial scaffolds (Harel, 2013; van Hiele, 1986). Mr. Drake's noticed this after his first implementation as he planned to spread out the lesson with more discussion "next time."

Discourse with AMS

AMS may directly contribute to students' identities as "received knowers" (Boaler & Greeno, 2000). Mr. Drake's students were given feedback on the assessment—not interpretation—of their inputs. Most, if not all, of the questions Mr. Drake's class saw asked for a short response in the form of a number or expression with a single correct answer. Current technology is not able to read and evaluate a natural language explanation. The hint-feedback and suggested video tutorials position learners as "received knowers," because students interacting with KA may only draw from provided information and then produce a simplified response. With the goal so narrow, there is little room for exploration or agency in the process of responding to a set of exercises. The knowledge is represented as something to adhere to: static videos and hints outlining the "correct" way to achieve the desired result: a right answer. Boaler and Greeno's "figured world" of received knowing is characterized by mathematical knowledge "transmitted to students, who learn by attending carefully to teachers' and textbook demonstration" (2000, p. 11). Mr. Drake's students working through the KA exercises were almost entirely attempting to adhere to procedure, "oh… you skipped one step. Oh, ok those numbers don't divide by x you cross multiply. Erase this part… keep the 2." The student conversation around the missing side of the triangle appeared as a troubleshooting session, as they negotiated the use of sine and cosine, by process of elimination because one wasn't working. The students expressed frustration, another element of the received knowledge world, "its too confusing, when to use sines and cos and tan."

The incentives for students, and indirectly for teachers, de-emphasize exploration of a concept in favor of adherence to procedures that yield correct responses. The reports shown to Mr. Drake told him which of his students had "achieved mastery", while the students were told to "get three in a row" correct to progress towards that mastery. The procedural conversations student to student and teacher to student reflected the incentives given by AMS, especially in contrast to the previous day's conceptual discussions. When helping students, Mr. Drake did not move beyond a procedural goal with the problems. He responded to specific points of confusion, but the conversations stopped when the student was able to input a correct answer—not necessarily meaning they had conceptual understanding.

The 'conversation' between student input and software feedback in KA is thin. When S3 talked with Mr. Drake about the triangle it was revealed she knew which sides were opposite, adjacent, hypotenuse (informally) but there were difficulties in connecting that knowledge with constructing a trigonometric equation and then solving it. KA is unaware of what idea in S3's head specifically makes the problem incorrect. Software that is purely assessment-based (no dynamic elements) is by definition unaware of the nature of the problem itself. KA does not interpret student inputs so can offer no specific feedback on an incorrect response.

Differences in Discourse AMS/DMS

The difference between KA and Desmos input modalities calls to mind Zevenbergen's (2000) example of a non-language based test item revealing similar math aptitude that lingual test items obscured. In Zevenbergen’s study, English language learners performed below their English-speaking peers on a test except on an item which communicated visually and less linguistically. KA’s exercises have higher pre-requisites to entry than Desmos Function Carnival. A student in KA must be able to read and parse the question including precise vocabulary. KA supports these skills with videos, but Mr. Drake had concerns that even then the videos were potentially linguistically challenging. Desmos’ Function Carnival has a minimum of written instructions and no numerical or computation requirements: but does have technological familiarity pre-requisites. It is very easy for students of all skills to press play and draw in a sketchpad—assuming trackpad or mouse dexterity and the recognition of the “play” button. One of the Mr. Drake’s groups, 6, may have confused the next button and then did not know how to proceed (or go back) thus leaving their first graph at its initial scribble for ten minutes. It is a non-useful assumption that young people are “good” with technology.

Function Carnival attempted to have certain concepts arise through the design of its questions and interaction. Cannon Man’s fall changes from quadratic to linear when his parachute opens. This not only allows two modeling tasks to be explored ‘at once,’ but allows a low-effort imprecise scribble animation more obviously deviate from the actual animation. A significant change in the portrayed situation corresponds to the significant feature of the Cartesian graph: “that dent” in a student’s words. Bumper Cars distance travelled graph is purposefully curved and oriented such that students who draw a mimicked ‘picture’ of the situation will produce a relation with two outputs from a single input. The likelihood of this topic arising is further supported by Desmos’ “sample student” who produces this exact kind of graph. Students analyzing and interpreting the work of others may encounter this issue, even if they themselves did not have it. Instead, they are tasked with explaining what must be fixed in such a graph. There are strong connections to Common Core's Standard for Mathematical Practice 3: "construct viable arguments and critique the reasoning of others" (Common Core State Standards Initiative, 2013). Furthermore, the bumper car travels at constant speed, potentially eliciting another Piagetian perturbation if students find disequilibrium in the idea a curved path can possibly have constantly increasing distance. These digital materials channel a pedagogy instilled in them by the authors, who have used some of the affordances of technology to elicit and support the development of certain concepts.

Function Carnival	KA Trigonometry
S: match the green car, so we're gonna stop where they hit, cause that's where he stops driving […] T: why does it getting hit mean the end? S: cause the distance will stop	S: I just divided T: ok by what. 27.5 by 11? What'd you get? S: 2.5 (T: ok) S: and I get the same with 30 and 12 (T: ok) S: so then I [inaudible] ohhhh I divided 19 by 2.5, I should have multiplied.
S1: ok so play it [they watch] S2: he'll be [waves hand] ya like here… constant speed S1: ok so right there he starts slowing down	S2: [hesitates] oh… you skipped one step. oh ok those numbers don't divide by x you cross multiply. Erase this part… keep the 2. So 2 times [inaudible] by itself right? Equals sin 50. x.

KA's trigonometry mission is not designed to promoting discussion, instead it is designed as a guide through topics paced via the student's ability to answer exercises. The exercises are checkpoints to assess if the student is able to perform a skill or apply a concept. KA is supporting a different part of Kysh's learning cycle: when a concept has been partially developed it should then be practiced and used to clear away "fuzziness" and make the concept more concrete (1999). The students above working in Function Carnival were exploring the potential concepts, like searching for Pickering's bridgeheads. In the above conversations, the students use imprecise or inaccurate language. The feedback from DMS structures their ability to keep exploring despite lacking entirely correct ideas. The students attempt transcriptions of their prior knowledge (when graphs "stop" or representations of speeds) and their attempts are not perfect. However, their thinking contains attempts at justifications and reasoning. The students above working in KA are attempting to apply procedures. These students are imprecise or inaccurate as well; the feedback from AMS only lets them know their results are inaccurate. These students hit a bump from AMS and are sent off to re-learn by drawing upon other resources like a hint, a video, peer or teacher help. The students are not exploring the concepts at hand, their thinking is limited to (imprecise) procedure.

A DMS activity for the scale ratio could show the student the resulting shape using their incorrect responses. The student may gain intuition from this feedback around why their answer is incorrect. An AMS exercise for distance and time graphs could ask closed questions (those with evaluable right answers) about definitions and procedures around function, constant speed vs. constant distance, or interpreting specific aspects of a graph. With an incorrect response, AMS could then suggest resources to define those ideas explicitly for the student, and collect data for a report to the teacher about which questions the student misses.

Chapter 6: Conclusion and Future Studies

DMS provides a bridge between concrete manipulatives and rigorous abstraction. In a flexible classroom, DMS can be the tool that creates virtual manipulatives to occupy-- and help to close-- the gap between the concrete and the abstract and enable visualization and conceptual understanding. Those virtual manipulatives can provide stepping-stones or talking points for students to discuss as they move up the van Hiele levels. But while the possibilities are vast, new doors are appearing faster than implementation strategies can open them. With both physical and computerized manipulatives, teachers have been urged to become knowledgeable in order to improve student attitudes towards math and conceptual skills (Clements & McMillen, 1996). A classroom structured after the recommendations of Horn (2008), may find virtual manipulatives to be useful as environments to display varied kinds of mathematical competence. But, the teacher's role is still vital, "simply using manipulatives is not enough if we do not consider *how* classroom teachers are using them." (Moyer, 2001) And the technological literacy required of teachers for sophisticated implementation of virtual manipulatives may be a large barrier even as access to computers rises. This barrier may be an opportunity to engage teachers with professional development with the specific goal of implementing DMS as virtual manipulatives and as a cohesive lesson.

Assessment Math Software (AMS) has a different role than Dynamic Math Software (DMS). DMS supports initial concept building by promoting exploration, agency and discourse. However, much is left in the teacher's hands in order to capture the potential from a DMS lesson. AMS supports procedure drilling and testing skills for the student, while offering data on the results to the teacher. DMS may best be used as a tool inside the classroom, where the teacher can scaffold discussions and guide students to achieving conceptual intuition. AMS may best be used as a support outside of the classroom since it is possible for students to work individually to practice procedures. DMS allows the well-prepared teacher to occasion multiple ways to be successful along multiple topics, but requires significant teacher preparation to take full advantage. AMS often has a specific way to be successful, but it can be useful if developing that specific method is a goal for the teacher or student. DMS is useful to support beginning to learn a concept, AMS is useful to support and provide practice with newly developed mathematics. Curriculum development and pedagogical practice must consider the comparative advantages of any software for the classroom. It is not enough merely to use technology in education. Like every other factor in teaching and learning, technology must be used with measured intent.

The Future

This study was small and brief and therefore we cannot draw conclusions about the value of DMS vs. AMS over the long term or even a semester. Within the results there are perhaps leads on how students conceptualize co-variational reasoning (like Johnson) with the aid of feedback from DMS. Future studies may focus more sharply on single elements of Function Carnival. For example, studying student learning using DMS graphs in which time is not a variable as Johnson (2016) recommends, or exploring the factors that lead to the split between Mr. Drake’s students who plotted the piecewise joint and those who did not. In Mr. Drake’s and Ms. Edward’s classes, some students Exploring the factors involved in students producing the piecewise joint or not. With preparation for audio and screencapture recording, a study could analyze student discourse in much greater detail and tie to specific interactions with Function Carnival or another DMS activity. This study was also only of one single DMS activity with a primary input method of graph sketching. Other common DMS modalities will also require closer examination.

There is room to explore differing modes of AMS and the effect of feedback types and timings on progression through the AMS itself. Especially if AMS becomes more sophisticated in the capability to diagnose a student's thought process. An alternative feedback timing would allow more questions to go by before returning assessment results to the student and offering hints or videos. Other AMS may perform deeper analysis on a set of responses to attempt a more precise diagnosis of the student's thinking.

Currently, KA offers sets of exercises from elementary to secondary to post-secondary topics in mathematics and other subjects: KA offers broad and a coherent program. Desmos currently offers seven[10] custom-designed, curated Activities such as Function Carnival and Marbleslides that have capabilities not present in the standard Desmos Calculator. But in addition, Desmos offers a framework and hosting for teachers to create Desmos Activities of their own[11]. These activities may use the standard calculator, collect aggregate graphs, ask and receive answers to questions[12]. At the time of this writing there are over 52,000 publicly created activities, 287 of which Desmos has "curated" to display more prominently[13]. GeoGebra also offers a sharing site[14] for publically created applets, currently hosting over 416,000 materials. Desmos and GeoGebra potentially offer breadth. The vast number of materials on both Desmos and GeoGebra are not cohesively ordered into a curricular plan like KA. However, the most carefully designed activities or applets, such as those custom authored "By Desmos" offer possibilities for depth like this study reveals.

[10] Seventeen if including the multiple varieties of Marbleslides and Polygraph
[11] https://teacher.desmos.com/
[12] The "Polygraph" is another framework the public may use to create custom activities
[13] https://twitter.com/Desmos/status/726109660084752384
[14] http://tube.geogebra.org/

As DMS and AMS will both continue to develop as computer capabilities increase and costs decrease. There will be a huge need for professional development specifically addressing pedagogical moves that relate to technological implementation. This study is my addition to NCTM's call in *Principles to Actions* stating that technology in the math classroom is misrepresented by smart boards and presentation software (Leinwand et al., 2014). A third category of technology, "Logistical Education Technology" might better fit those products whose focus is improving non-content and non-pedagogical areas of teaching. Clements, Sarama, Yelland, & Glass (2008) say policymakers and curriculum developers miss the potential for revolutionary pedagogies. The problem is a few layers deep: "what we need to see are ways in which curricula can be re-conceptualized to illustrate the ways in which they can modified to reap the benefits of new technological environments for learning and teaching geometry." NCTM, six years later, calls for special consideration and study of the pedagogies and practices involved in technology based lessons. I join them in looking forward to pedagogical principles that transcend specific digital products to become good practice in general.

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Appendix A: Tables

Table 1

1 Coded Summary of Graphical Responses

Group ID	Cannon Man	Bumper Cars	Ferris Wheel	comments
G1	F P 1	F 2 *	F 2 *	
G2	L 2 *	L 2 *	L 2 *	
G3	F 1	F 1	F 1 *	
G4	F 1	F P 1 *	F 2 *	Described using notches on x axis to place time slider, then make a mark.
G5	F 1	F 1 *	F 1	
G6	F 0	F 0	F 1	Non participant in #1, engaged on 2 and 3, but left 2 in an inaccurate state
G7	L 2 *	L 2 *	L 2 *	
G8	F 0	F 2	F 1	
G9	F 2 *	F 2 *	F 2 *	Freehand is in small sections, could have used notches
G10	F 2 *	P 1	P 2 *	Appeared to use freehand or points at intervals, not necessarily corresponding to notches
G11	L P 1 *	L 2	F 1	Switched to Freehand tool on Ferris Wheel
G12	L 0	L 2	L 2 *	
G13	L 1	L 2†	L 1	†Dropped distance to 0 on Bumper cars at crash

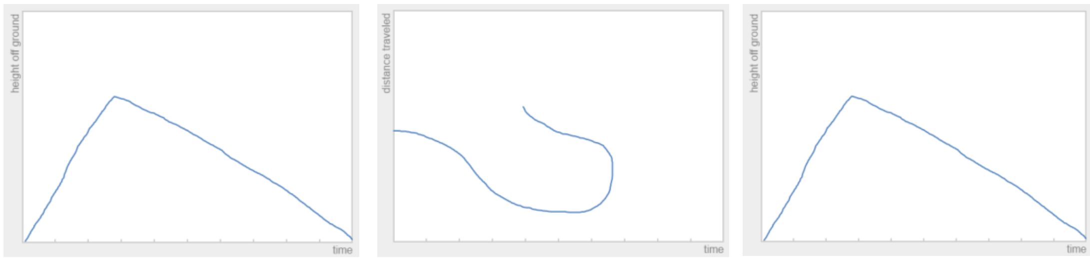
Note: F, P and L denote if the Freehand, Point, and Line Segment tools were evident in the final product of the group's graph. A score of 2 means the graph matched the animation very precisely, a score of 1 means the graph mostly matched the animation, and 0 means the graph had large sections that did not match the animation. The symbol (*) denotes the existence of a particular graphical feature emphasized by the teacher in later discussions. In Cannon Man: the piecewise joint between parabola and line, in Bumper Cars: the horizontal line after the crash, in Ferris Wheel: symmetry of the graph.

Table 2

2 Summary of Graphical Features for the 13 final product graphs

	Feature	Count for Cannon Man	Count for Bumper Cars	Count for Ferris Wheel
Tool usage	F – Freehand	8	7	8
	P – Point	2	2	1
	L – Line segment	5	5	4
Rubric Distribution	2 – precise match	4	8	7
	1 – match	6	4	6
	0 – no match	3	1	0
	* – graph includes emphasized feature	5	6	8

Table 3



Cannon Man: parabola/line joint, Bumper Cars: path instead of distance, Ferris Wheel: lines instead of curves, symmetry

3 Group Written Responses to Desmos' Graph Interpretation Questions

Group ID	Cannon Man		Bumper Cars		Ferris Wheel	
G1	He didn't draw the height of the cannon man as high as it should be		It states that the car started in the middle went further then at one point the car stopped and the time did as well .to fix this she should try and make a straighter line		Her lines are too steep and sharp, and the wheel is moving at the same speed the whole time so they should all have same incline/decline	*
G2	Cannon man is not reaching the top at a quicker speed	r	Jenny followed the trail of the bumper car instead of tracing the distance of the car has traveled. In her graph, you can see that the car goes backwards. This does not correlate with the distance it has traveled.	*	Her graph does not align with the way the cart travels. Sophia's graph travels rapidly without any curves.	* r
G3	Eric's graph predicts that the cannon man will gradually fall to the surface . In order to fix his graph he would have to change decrease the time that the cannon man will fall.	* r	Jenny's graph tells us the course of the driver in the bumper car over the amount of time . To help Jenny fix her graph we woud suggest that she would give more information on the speed of the bumper car to determine further information.	* r	Predicting on Sofia's graph he green cart will rapidly descend and slowly go back up . To fix her graph we will have to change the distance and time intervals.	r
G4	He should have represented the amount of time it took to land.	*	She says that the bumper car will stop. She should make the distance and time seem more accurate.		She says that the cart is going down faster and then going back up slower. She should show the graph slowing down when the cart goes down and continuing at the same speed.	r
G5	As the cannon blows him up , he will have a constant speed going downwards . Eric could try to match it up with the speed of the cannon man.	* r	We started at 0 and worked our way up . when the car gets hit , the car distance will stop and the graph should be a straight line.			
G6	The graph clearly shows that the cannon man's height rises however, once it reaches his highest point, he begins to fall. To help him, you can tell him that the cannon man also has a parachute which helps him fall slowly.	r	The bumper car will be going backward at one point. To help her, she can have a more accurate line.	*		

G7	Eric's graph say that the cannon man goes high and drops at a steady passe [pace], but it does not show any details in were the cannon man is slowing down in the middle of the air and when he lets his parachute out.	* r	Her graph shows the line where the car traveled when the graph is about the distance and time the car traveled	*	Sofia's graph show the cart is moving at a fast rate when it goes down , then when it goes back up is a slower rate. The graph moves at a smoother and regular passe. [pace]	r
G8	Eric's graph show that the cannon man will start slowly and come even slower. He can fix his graph by making it steeper.	r	In Jenny's graph it shows that the bumper car would most likely go backwards because of the curve on the line. Also it does follow the line of the bumper car.	*		
G9	The cannon man went into the air really fast and descended really fast. Eric did now show where the cannon man slowed down.	* r	She drew the direction that the car drove and not the correct distance.	*	The cart goes up and down really fast. She should fix the second cycle of the ferris wheel	* r
G10	The graph states that there is amplitude or a reaching point. What he could fix is the slope of the parachute	*	Jenny's graph states a constant path but will soon create a car crash where the movement stays still. She would need to fix the constant of the distance over the time of the car crash		The graph states the amplitutde of the cycle of the ferris wheel. She should fix the period of the second cycle highest reaching point.	*
G11	Cannon man goes higher in the graph,then he slows down and your graph shows a constant move of him however it takes him longer to touch the floor because of his para shoot	*	This is just the outline of the race track goes this does not include the right time because time goes back in her graph	*	Her graph shows the caert going out if the wheel she needs to give a curve.	*
G12	Eric's graph for cannon man shows that in the first couple of minutes Cannon Man gradually increased his height off the ground then at the peak it shows Cannon Man decreasing rapidly. To help this graph Eric should put a divot in the graph to slow down the	*	Jenny's graph says that it will be the same as the regular clip. I will say that to use graph to help her instead of just right away.	*		
G13	Eric's graph shows the cannon man rapidly shot from the cannon at a fast speed which increases his height but overtime his height decreases at a slow rate as time passes by.	r	Jenny's Graph shows that the bumper car traveled at a slow rate, then increased its distance but then made a sudden stop.	r	Sophia's graph has sharp lines that indicate that the cart gradually goes up but as it turns over to the other side the graph she roughly sketched would go down back to where the cart started.	* r

The symbol (*) indicates the response is relevant to a main issue with the given graphs, the code 'r' notes responses that refer to speed or rate.

Table 4

4 Count of Mr. Drake's students logging into Khan Academy over time

logged into KA January - April 2016		
Student	Tuesdays	other days
1	7	11
2	6	7
3	6	20
4	4	0
5	5	0
6	5	0
7	5	4
8	4	0
9	6	5
10	1	1
11	6	7
12	4	1
13	4	0
14	5	0
15	5	7
16	5	0
17	5	0
18	3	1
19	5	2
20	6	1
21	5	0
22	5	3
23	3	1
24	3	1
25	4	1
median	5	1
mean	4.68	2.92

Appendix B: Observation Protocol

Observations at minute		5	10	15	20	25	30	35	40	45	Notes Teacher Name: Date:
Teacher's Activity	1. Managing behavior/materials 2. Direct instruct. Lecturing 3. Procedure / 'helping' 4. Probing/Questioning to promote progress 5. Questioning to extend concept	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	
Teacher role / T interacting with…	1. Whole class 2. Small groups 3. individuals 4. (observing/making notes)	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	
Student Cognitive Activity	1. Receipt of knowledge 2. Applying procedural knowledge 3. Addressing perturbations / feedback 4. Knowledge construction, synthesis	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	
Student Talk	1. Off Mostly Off task or no talking at all 2. About Logistics of lesson/software/hardware 3. About accuracy / grading / status 4. In concept procedures, definitions 5. With reasoning ("why"/ "because") 6. Beyond conjecture, generalization, "what if"	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	
Student reaction to perturbation or indication of error	1. Retries 2. Looks for direct help in software 3. Asks for help 4. Disengages	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	
Student Exploration of software	1. Only base functionality used 2. One or more optional element tried 3. One or more optional elements used moderately 4. Frequent use of optional elements 5. Goes beyond.	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	
Other	1. 2. 3. 4. 5.	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	

Appendix C: Additional Figures

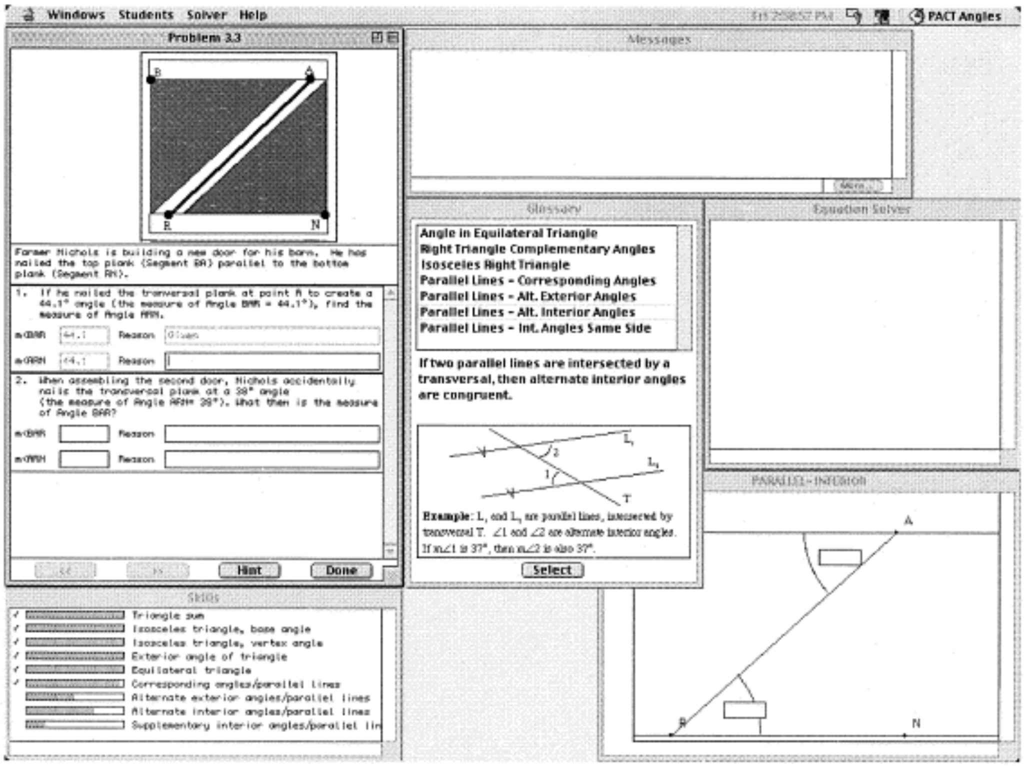


Figure 15 A Cognitive Tutor adaptation from 2001 that allowed students to select reasons from a Glossary to justify answers.(Aleven & Koedinger, 2002)

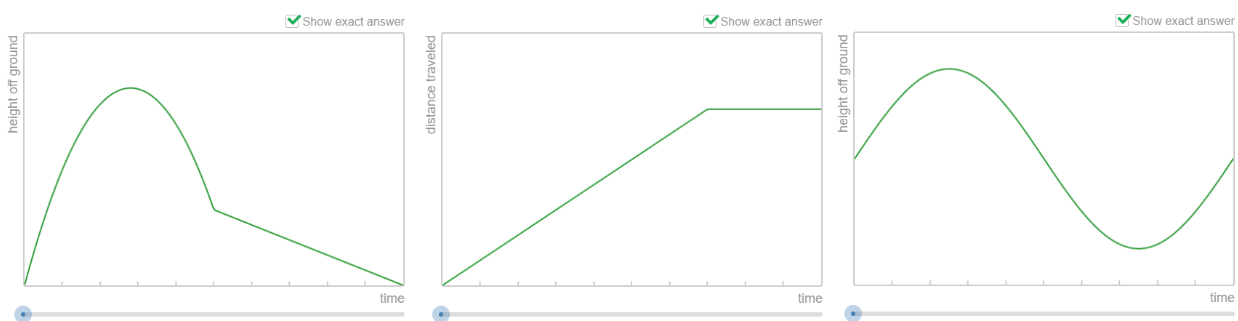


Figure 16 Graphs that exactly match the animation for Cannon Man, Bumper Cars, Ferris Wheel